The Blow-up Lemma and growing degrees

Andreas Würfl

Technische Universität München

Combinatorics Conference in Lisbon

(joint work with Julia Böttcher, Yoshi Kohayakawa & Anusch Taraz)

1 The Blow-up Lemma

2 Growing degrees
- Arrangeability of graphs
- Random greedy embedding

3 Making things rigorous
- Auxiliary graphs
- Matchings in auxiliary graphs

Definition

(ε, δ)-SUPER-REGULAR

Let $\varepsilon, \delta > 0$. The graph $(A \dot\cup B, E)$ with $|A| = |B| = n$ is an (ε, δ)-*super-regular* pair if

- $|d(X, Y) - d(A, B)| \leq \varepsilon$ for all $X \subseteq A$, $Y \subseteq B$ with $|X|, |Y| \geq \varepsilon n$,
- $\deg(v) \geq \delta n$ for all $v \in A \cup B$.

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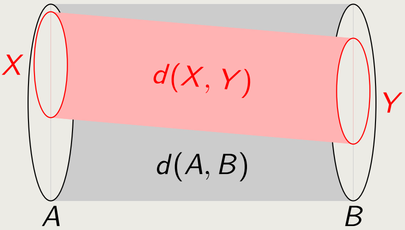

- regularity $\leftrightarrow$ "densities as expected"
- high minimum degree

The Blow-up Lemma

Komlós, Sárközy, Szemerédi, '97

For $\delta > 0$, $\Delta \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \Delta$ and $H \subseteq K_{n,n}$ then H is subgraph of any (ε, δ)-super-regular pair $G = (A \dot\cup B, E)$ with $|A| = |B| = n$.

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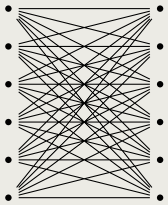
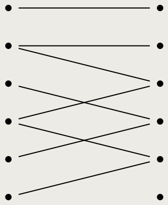
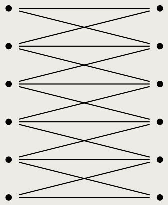


$K_{n,n}$



H



G

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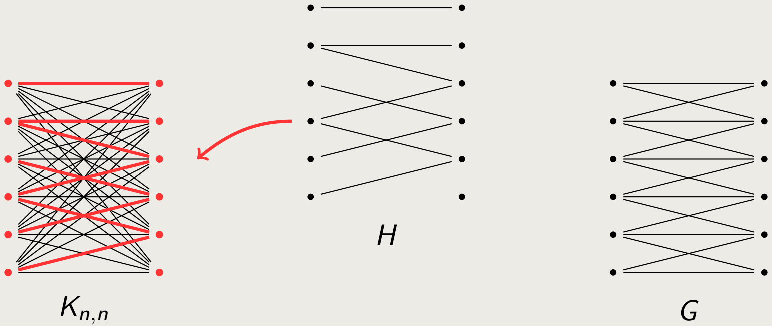

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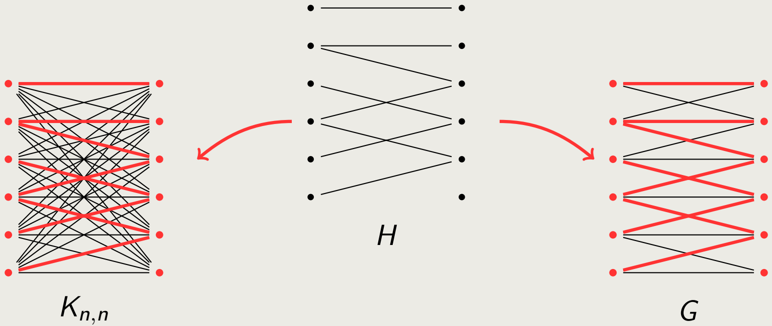

$H \subseteq K_{n,n} \quad \Rightarrow \quad H \subseteq G$

The Blow-up Lemma for growing degrees

Theorem Komlós, Sárközy, Szemerédi, '97

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Theorem Böttcher, Kohayakawa, Taraz, W. '11

For $\delta > 0$, $a \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \sqrt{n}/\log n$ and $H \subseteq K_{n,n}$ is a-arrangeable then H is subgraph of any (ε, δ)-super-regular pair $(A \dot\cup B, E)$ with $|A| = |B| = n$.

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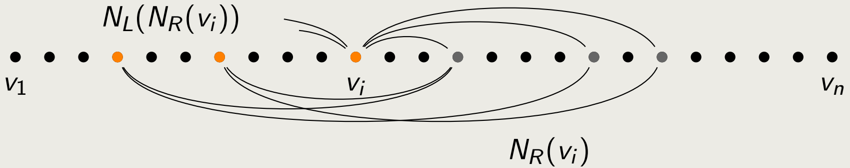
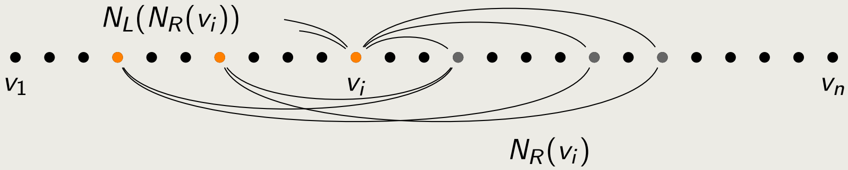
For $\delta > 0$, $a \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \sqrt{n}/\log n$ and $H \subseteq K_{n,n}$ is *a*-arrangeable then H is subgraph of any (ε, δ)-super-regular pair $(A \dot\cup B, E)$ with $|A| = |B| = n$.

$G = (V, E)$ is *a-arrangeable* if there is an ordering $V = \{v_1, \ldots, v_n\}$ with $|N_L(N_R(v_i))| \leq a$ for all $i = 1, \ldots, n$.

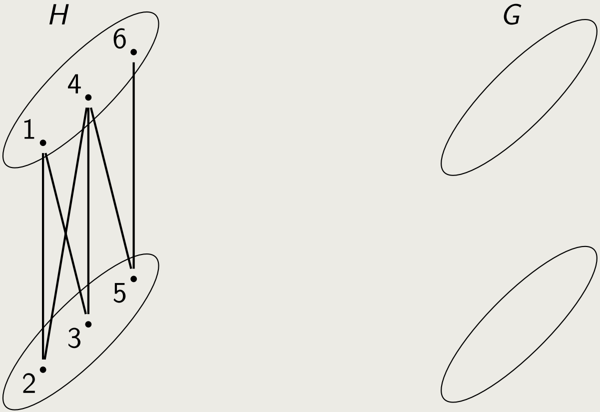


Examples of arrangeable graphs



- trees are 1-arrangeable
- planar graphs are 761-arrangeable — CHEN, SHELP '93
- planar graphs are 10-arrangeable — KIERSTEAD, TROTTER '94
- graphs without a K_p-subdivision are p^8-arrangeable — RÖDL, THOMAS '94

Sketch of proof: a random greedy embedding



(ε, δ)-super-regular

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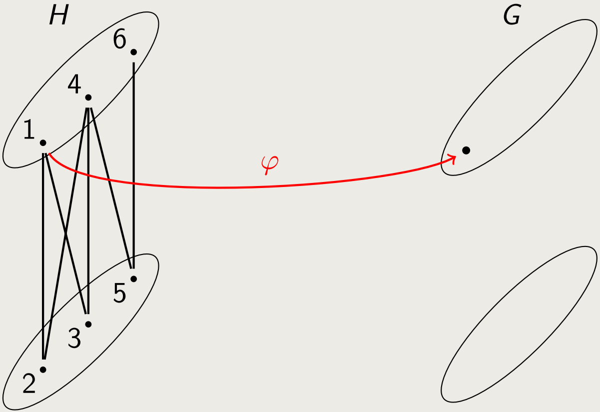

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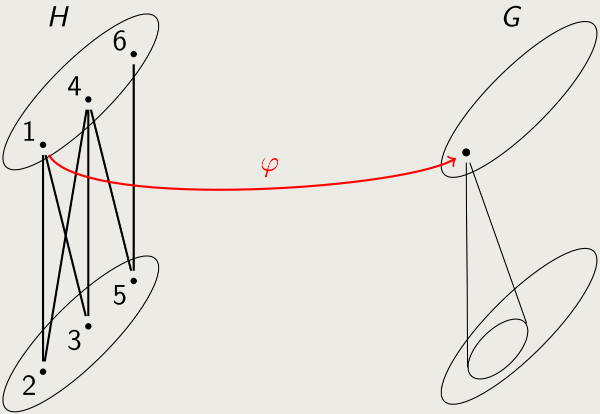

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$$C(x) = \bigcap_{y \in N_L(x)} N_G(\varphi(y))$$

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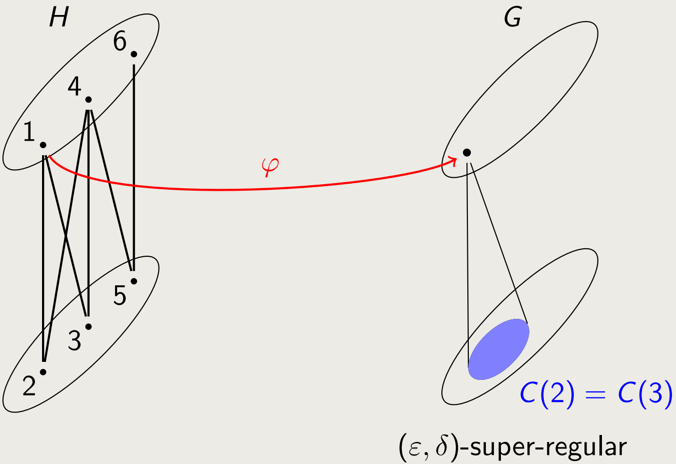

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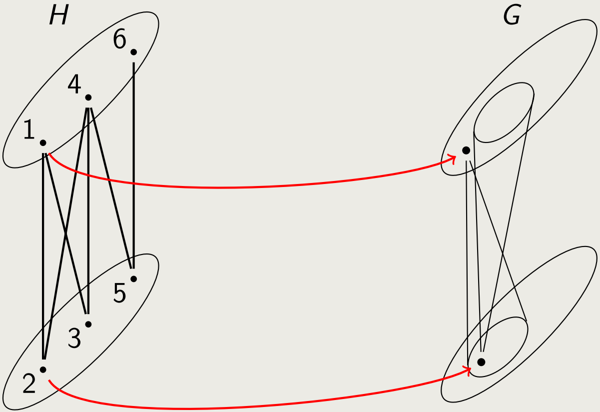

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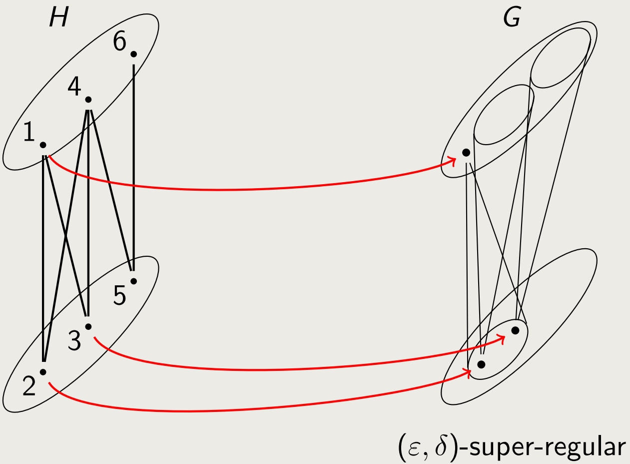

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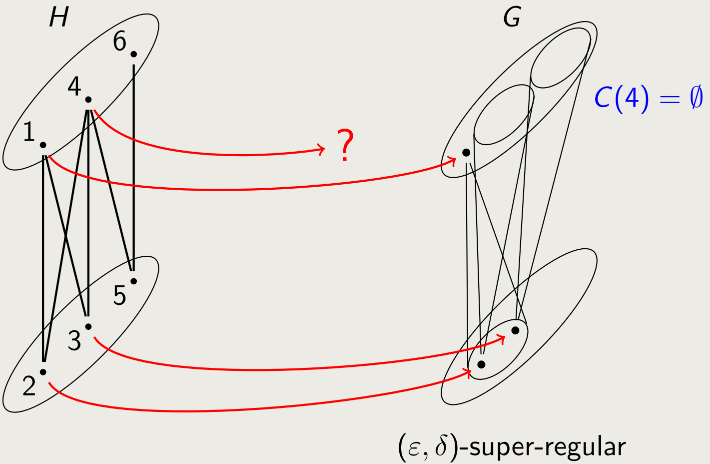

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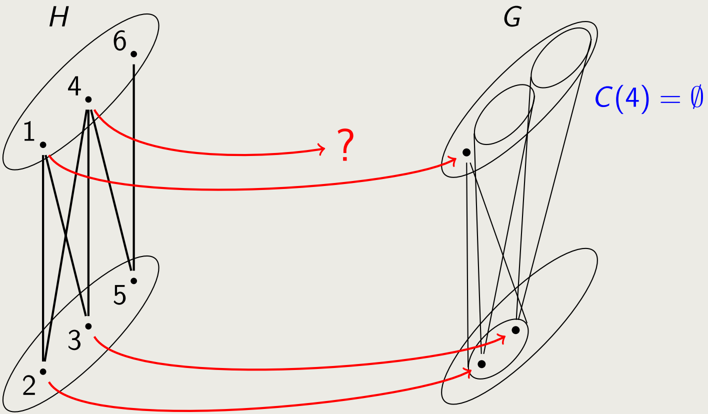

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MESSAGE: respect your successors!

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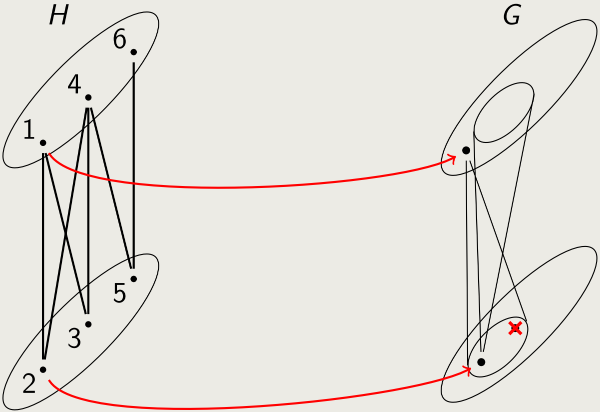

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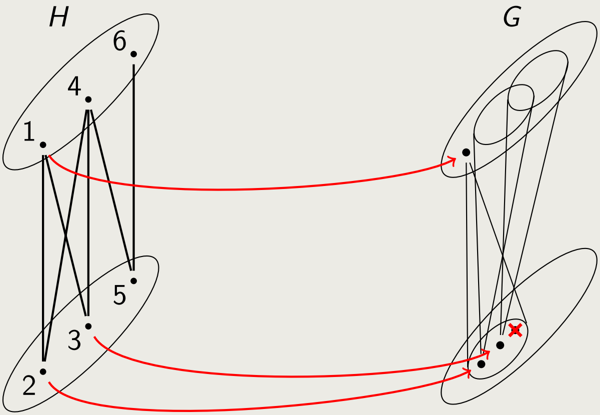

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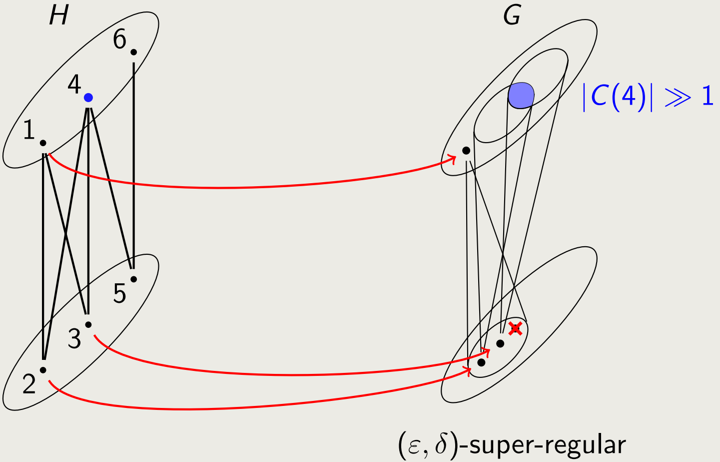

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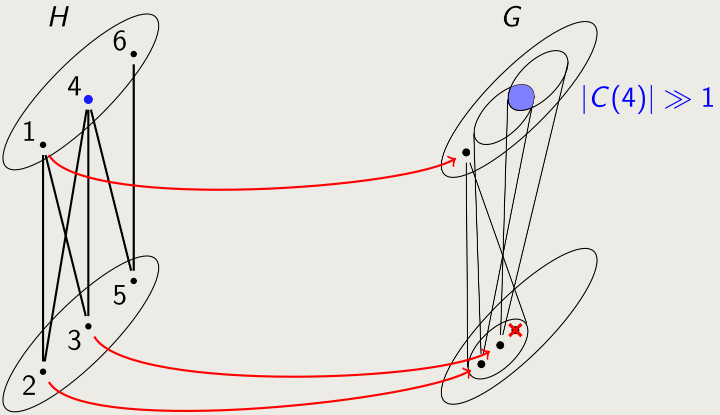
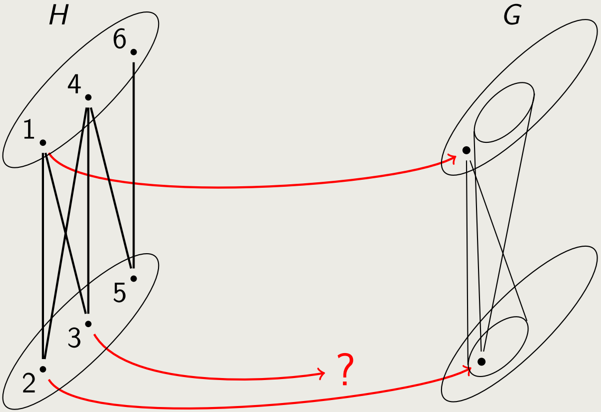


(ε, δ)-super-regular

Does this always work?

Respect one successor. . .

. . . with the help of ε-regularity



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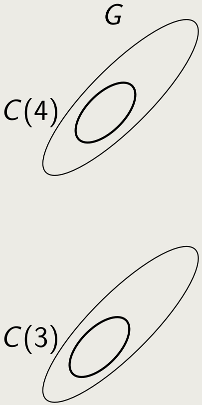

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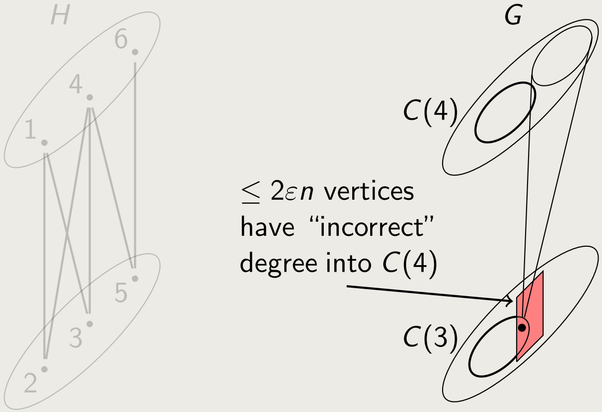

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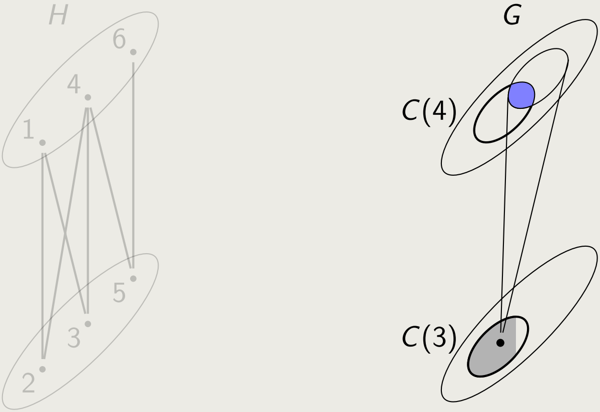

(ε, δ)-super-regular

all but $2\varepsilon n$ vertices in $C(3)$ have "correct" degree into $C(4)$

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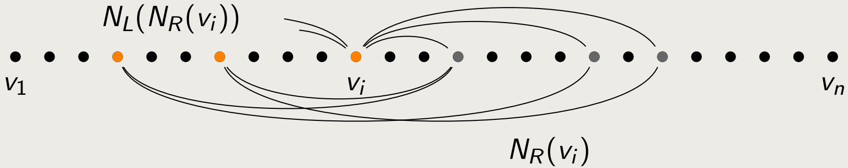

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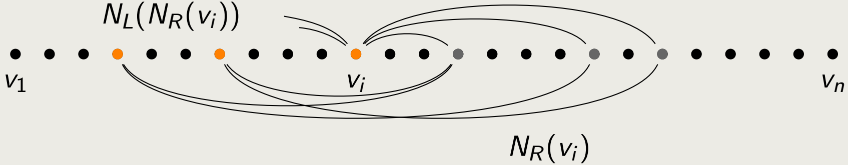
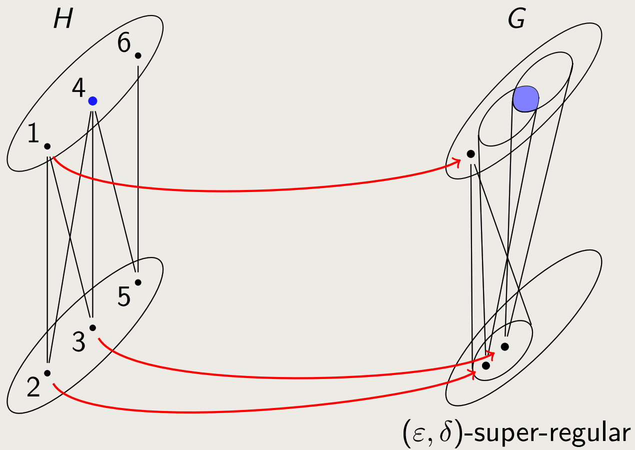


all successors of v_i have at most a predecessors in total

$\Rightarrow$ we have to respect at most 2^a *different* candidate sets

$\Rightarrow$ we loose at most $2^a \varepsilon' n$ candidates

The auxiliary graphs



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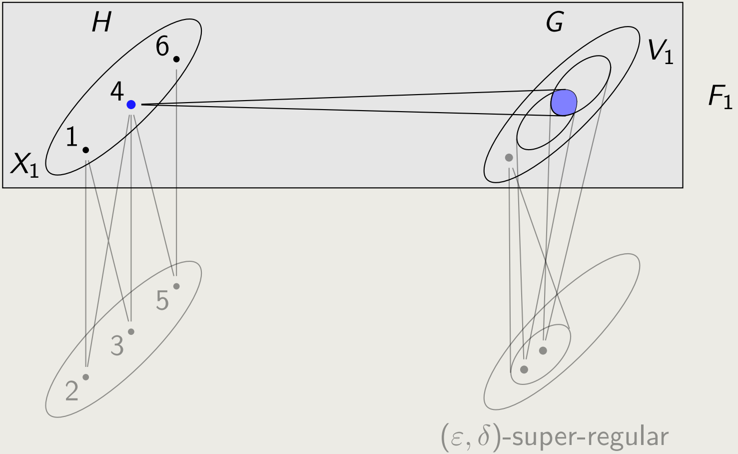

$$F_1 = (X_1 \cup V_1, E_1) \text{ with } \{x, v\} \in E_1 \text{ iff } v \in C(x)$$

One auxiliary graph, two nice properties

Lemma

With probability $1/2$ all auxiliary graphs F_i inherit *some* regularity from G.
$\Rightarrow$ random embedding algorithm gives an **almost spanning embedding**.

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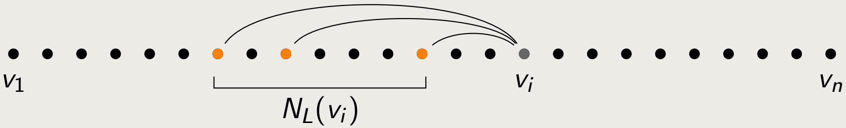
- $\exists$ perfect matching in the "remaining part" of each auxiliary graph
- these perfect machtings define a **spanning embedding**.

The 'obvious' next step

$G = (V, E)$ is *d-degenerate* if there is an ordering $V = \{v_1, \ldots, v_n\}$ with $|N_L(v_i)| \leq d$ for all $i = 1, \ldots, n$.



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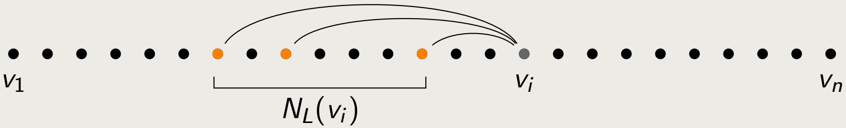

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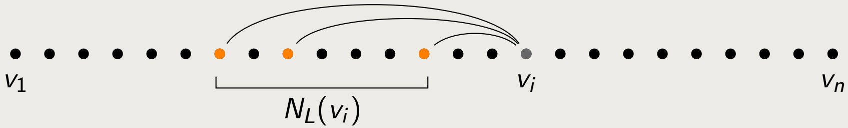

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Far goal

For $\delta, \mu > 0$, $d \in \mathbb{N}$ there is $\varepsilon, \xi > 0$ such that the following holds. If H has $\Delta(H) \leq \xi n$ and $H \subseteq K_{(1-\mu)n,(1-\mu)n}$ is d-degenerate then H is subgraph of any (ε, δ)-super-regular pair $(A \dot\cup B, E)$ with $|A| = |B| = n$.