

Induced C_5 -free graphs of fixed density: counting and homogenous sets

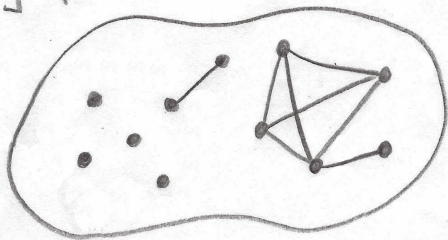
Andreas Würfl

Technische Universität München

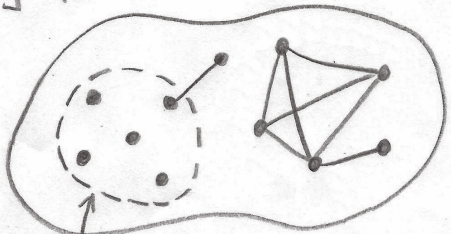
EuroComb 2011, Budapest

(joint work with Julia Böttcher & Anusch Taraz)

graph:

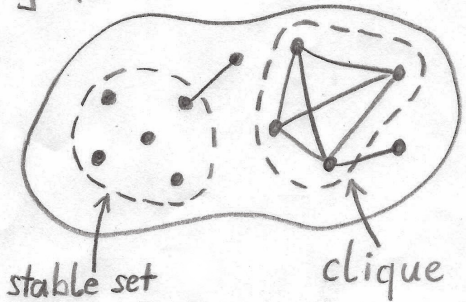


graph:

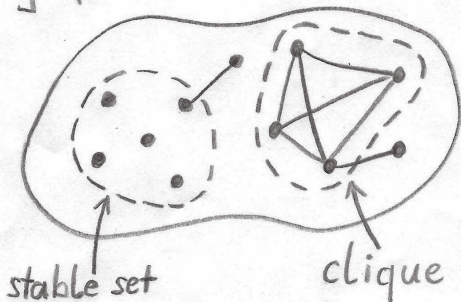


stable set

graph:



graph:

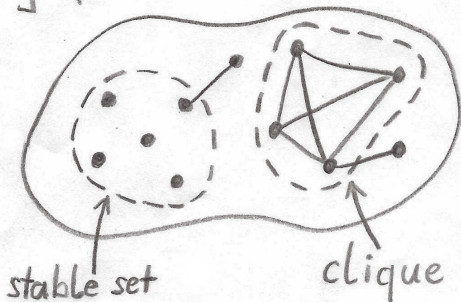


Ramsey [1930]

every graph has a
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of size $\frac{1}{2} \log n$



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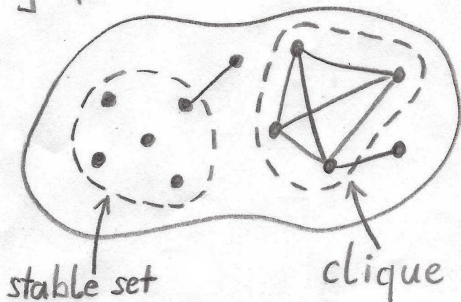


Erdős [1947]

there are graphs without a homogeneous set of size $2 \log n$



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Erdős, Hajnal [conjecture, 1989]

for every H there is $\varepsilon(H) > 0$ s.t. any graph without an induced copy of H has a homogeneous set of size $n^{\varepsilon(H)}$



For every graph H there is $\varepsilon(H) > 0$ s.t. all graphs $G \in \text{FORB}_n^*(H)$ have $\text{hom}(G) \geq n^{\varepsilon(H)}$.

$$\text{FORB}_n^*(H) = \{G : |G| = n, H \not\subseteq G\}, \quad \text{hom}(G) = \max\{\alpha(G), \omega(G)\}$$

Conjecture

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The existence of some $\varepsilon(H) > 0$ is known, e.g., for

$$H \in \{P_3, K_3, P_4, K_{1,3}, C_4, K_4 - e, K_4, \text{Bull}\}.$$



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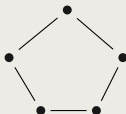
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What about $H = C_5$?



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For every graph H there is $\varepsilon(H) > 0$ s.t. **almost all** graphs $G \in \text{FORB}_n^*(H)$ have $\text{hom}(G) \geq n^{\varepsilon(H)}$.

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They ask:

For which H is there $\eta(H) > 0$ s.t. almost all graphs $G \in \text{FORB}_n^*(H)$ have $\text{hom}(G) \geq \eta(H) \cdot n$?

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Example:

every graph in $\text{FORB}_n^*(P_3)$ is the disjoint union of cliques

- $\text{FORB}_n^*(P_3)$ is dominated by graphs G with $\omega(G) = \Theta(\log n)$
- thus P_3 does **not** have any $\eta(P_3) > 0$

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They say:

“ P_3 does not have any $\eta(P_3) > 0$ ”

“ P_4 might not have any $\eta(P_4) > 0$ ”

“we wonder if all other H have some $\eta(H) > 0$ ”

$H = C_5$ has the property even in a **stricter sense**:

Theorem

BÖTTCHER, TARAZ, W, 2010

For every rational $c \in (0, 1)$ there exists $\eta > 0$ s.t. almost all graphs $G \in \text{FORB}_n^*(C_5, c)$ have $\text{hom}(G) \geq \eta n$.

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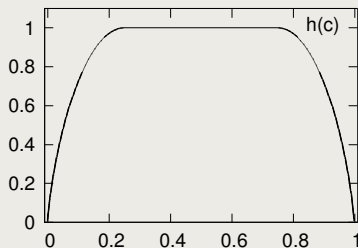
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 - basic combinatorics

Define

$$h(c) := \begin{cases} H(2c)/2 & \text{if } 0 < c < \frac{1}{4}, \\ 1/2 & \text{if } \frac{1}{4} \leq c \leq \frac{3}{4}, \\ H(2c-1)/2 & \text{if } \frac{3}{4} < c < 1. \end{cases}$$

Here $H(c)$ is the *binary entropy*

$$H(c) = -c \log c - (1-c) \log(1-c).$$



Binomial coefficients

For every $c \in (0, 1)$ with cn being an integer

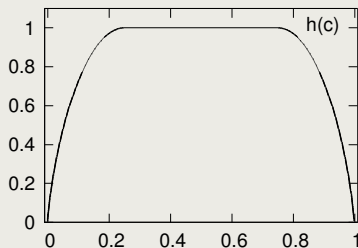
$$\binom{n}{cn} = 2^{H(c)n + o(n)}.$$

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Lemma

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$$|\text{FORB}_n^*(C_5, c)| \geq 2^{h(c) \binom{n}{2} + o(1) \binom{n}{2}}.$$

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The aim of Step 2:

Lemma

For every $c \in (0, 1)$ with $c \binom{n}{2}$ being an integer there is $\eta > 0$ s.t.

$$|\text{FORB}_{n,\eta}^*(C_5, c)| \leq 2^{(h(c) - r(c)) \binom{n}{2} + o(1) \binom{n}{2}},$$

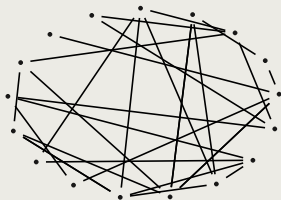
where $r : (0, 1) \rightarrow \mathbb{R}^+$.

$$\text{FORB}_{n,\eta}^*(C_5, c) = \{G \in \text{FORB}_n^*(C_5, c) : \text{hom}(G) < \eta n\}$$

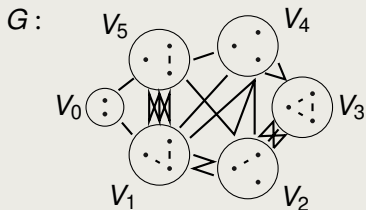
Every sufficiently large graph $G = (V, E)$ can be equipartitioned into $V = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_k$ s.t. almost all pairs (V_i, V_j) are ε -regular pairs.

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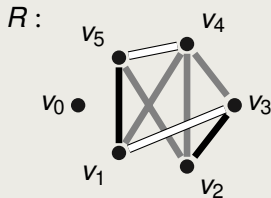
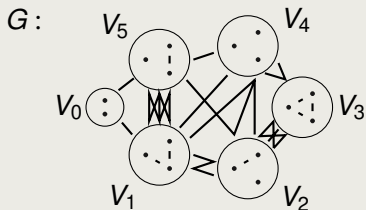
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Then G has the **coloured reduced graph** $R = (\{v_0, \dots, v_k\}, E_R, \sigma)$ with

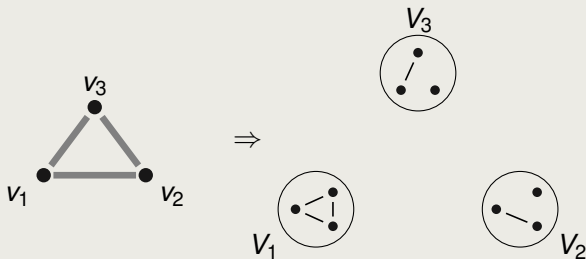
$$\sigma : E_R \rightarrow \{\blacksquare, \square, \square\}, \quad \{v_i, v_j\} \mapsto \begin{cases} \blacksquare & \text{if } d(V_i, V_j) > (1 - d) \\ \square & \text{if } d(V_i, V_j) \in [d, 1 - d] \\ \square & \text{if } d(V_i, V_j) < d. \end{cases}$$

The mysterious regular pairs

ε -regular pairs allow the embedding of (fixed) bipartite graphs
especially: any triangle of ε -regular pairs gives an induced C_5

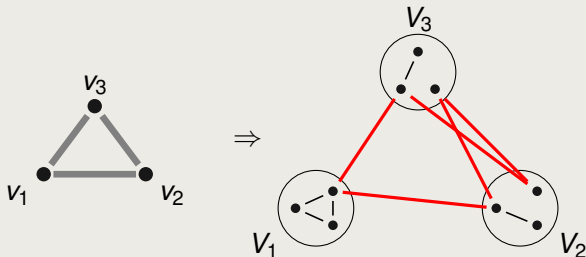
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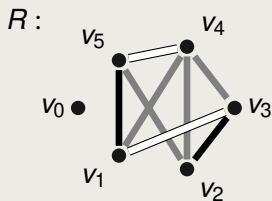
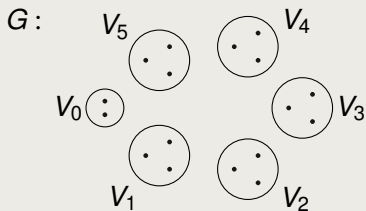
The structure of R

Let $R = (\{v_0, v_1, \dots, v_k\}, E_R)$ be the coloured reduced graph of some $G \in \text{FORB}_n^*(C_5)$. Then at most $k^2/4$ edges in R are grey by Turán's Theorem.

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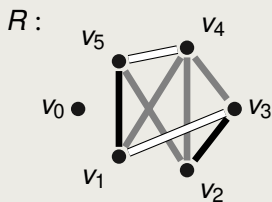
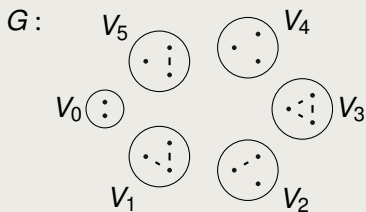
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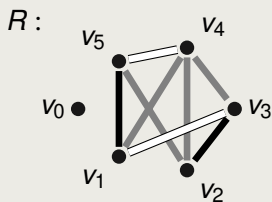
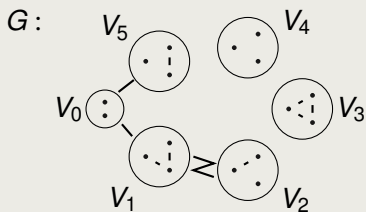
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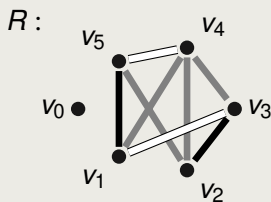
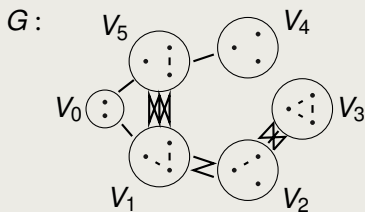
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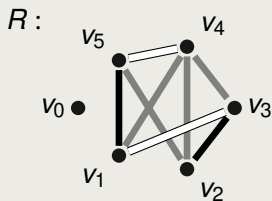
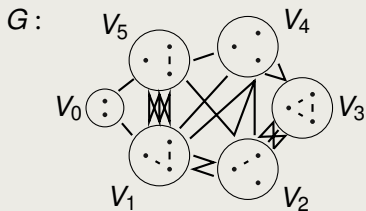
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- edges into **grey edges** of R .

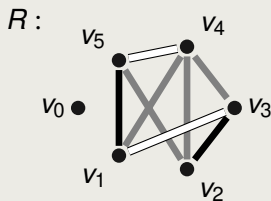
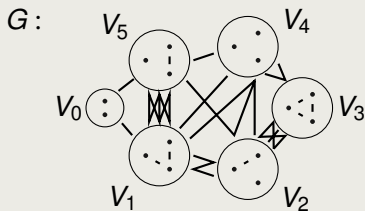


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negligible
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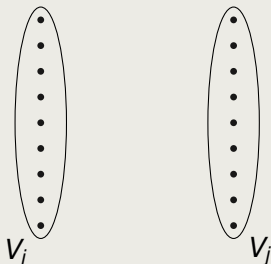
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G :



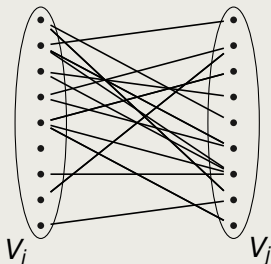
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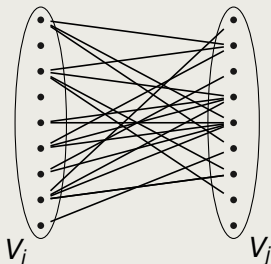
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this gives no more than $2^{h(c)\binom{n}{2}+o(n^2)}$ choices

R :



G :



Lemma

Let G be a graph on n vertices. Then one of the following is true:

- there is a homogenous set of size $n/6$ in G , or
- G has $n/6$ disjoint copies of P_3 , or
- G has $n/6$ disjoint copies of $\overline{P_3}$.

Lemma

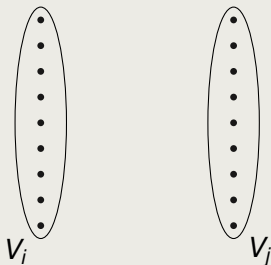
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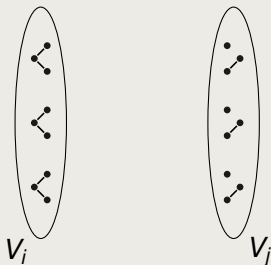
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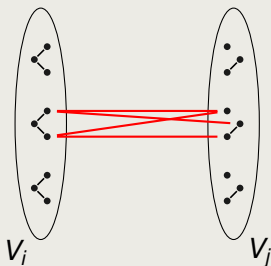
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some choices are forbidden!

Lemma

There is $r(c) > 0$ s.t. at most

$$2^{(h(c)-r(c))\binom{n}{2}+o(1)\binom{n}{2}}$$

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Summing up:

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RAMSEY

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almost all $G \in \text{FORB}_n^*(C_5)$ have **linear** homogeneous sets,
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