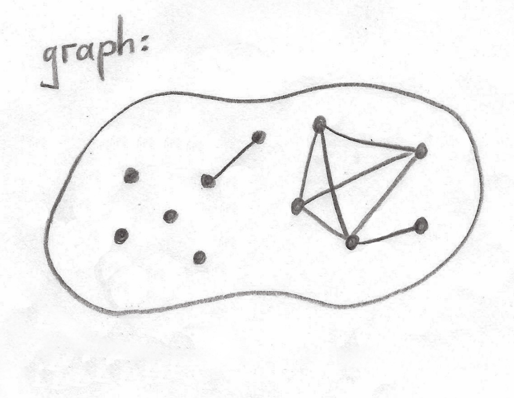
Induced C_5-free graphs of fixed density: counting and homogenous sets

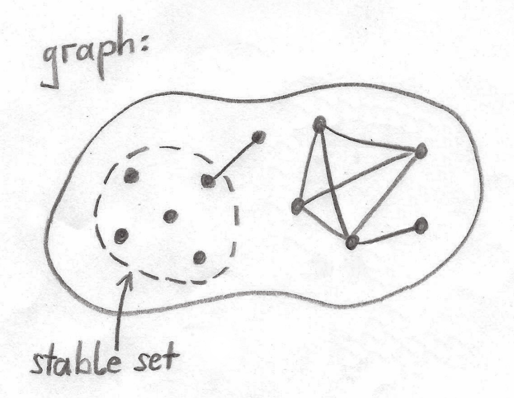
Andreas Würfl

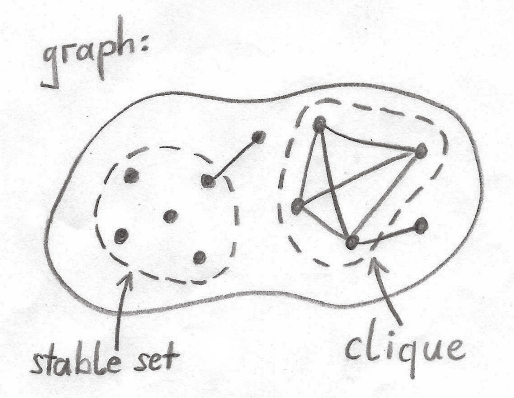
Technische Universität München

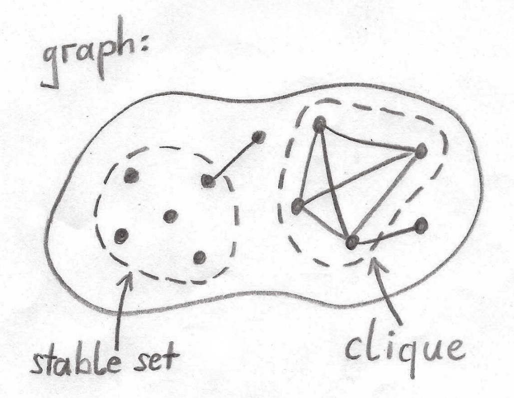
EuroComb 2011, Budapest

(joint work with Julia Böttcher & Anusch Taraz)


graph:


graph:
stable set


graph:
stable set
clique



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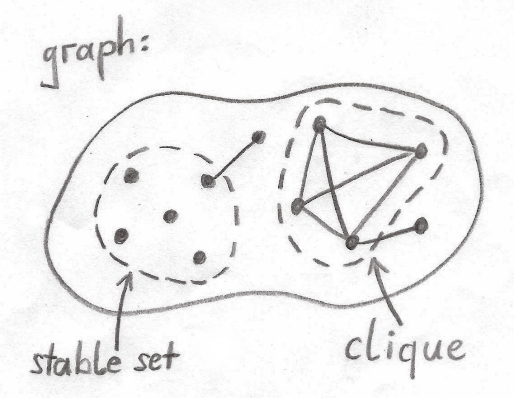
every graph has a homogeneous set of size $\frac{1}{2} \log n$



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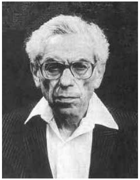

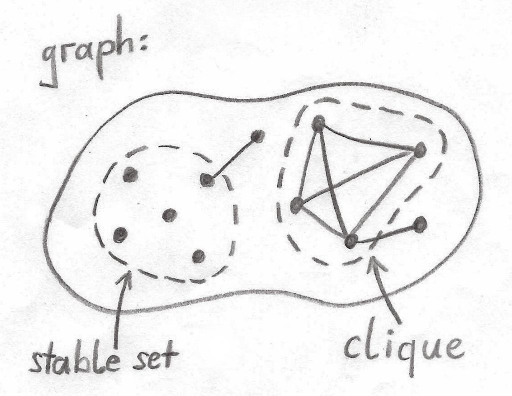

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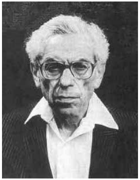

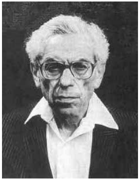

Erdős, Hajnal [conjecture, 1989]

for every H there is $\varepsilon(H) > 0$ s.t. any graph without an induced copy of H has a homogeneous set of size $n^{\varepsilon(H)}$

Conjecture

Erdős, Hajnal 1989

For every graph H there is $\varepsilon(H) > 0$ s.t. all graphs $G \in \textsc{Forb}_n^*(H)$ have $\hom(G) \geqslant n^{\varepsilon(H)}$.

$\textsc{Forb}_n^*(H) = \{G : |G| = n, H \not\leq G\}$, $\qquad \hom(G) = \max\{\alpha(G), \omega(G)\}$

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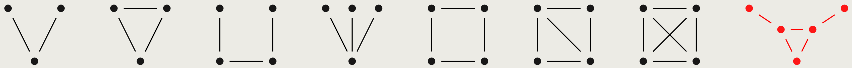
The existence of some $\varepsilon(H) > 0$ is known, e.g., for

$$H \in \{P_3,\ K_3,\ P_4,\ K_{1,3},\ C_4,\ K_4 - e,\ K_4,\ \text{Bull}\}.$$



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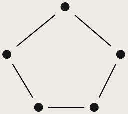
What about $H = C_5$?



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They ask:

For which H is there $\eta(H) > 0$ s.t. almost all graphs $G \in \textsc{Forb}_n^*(H)$ have $\hom(G) \geqslant \eta(H) \cdot n$?

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Example:

every graph in $\textsc{Forb}_n^*(P_3)$ is the disjoint union of cliques

- $\textsc{Forb}_n^*(P_3)$ is dominated by graphs G with $\omega(G) = \Theta(\log n)$
- thus P_3 does not have any $\eta(P_3) > 0$

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They say:

"P_3 does not have any $\eta(P_3) > 0$"

"P_4 might not have any $\eta(P_4) > 0$"

"we wonder if all other H have some $\eta(H) > 0$"

$H = C_5$ has *the property* even in a stricter sense:

Theorem

BÖTTCHER, TARAZ, W, 2010

For every rational $c \in (0, 1)$ there exists $\eta > 0$ s.t. almost all graphs $G \in \text{FORB}_n^*(C_5, c)$ have $\hom(G) \geqslant \eta n$.

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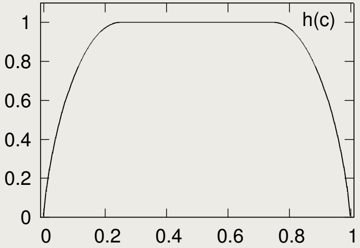
   - Turán's Theorem
   - basic combinatorics

Define

$$h(c) := \begin{cases} H(2c)/2 & \text{if } 0 < c < \frac{1}{4}\,, \\ 1/2 & \text{if } \frac{1}{4} \leqslant c \leqslant \frac{3}{4}\,, \\ H(2c-1)/2 & \text{if } \frac{3}{4} < c < 1\,. \end{cases}$$



Here $H(c)$ is the *binary entropy*
$H(c) = -c\log c - (1-c)\log(1-c)$.

Binomial coefficients

For every $c \in (0, 1)$ with cn being an integer

$$\binom{n}{cn} = 2^{H(c)n+o(n)}.$$

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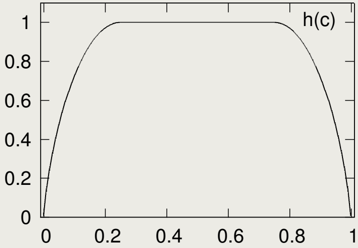

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Lemma

For every $c \in (0,1)$ with $c\binom{n}{2}$ being an integer

$$|\textsc{Forb}_n^*(C_5, c)| \geqslant 2^{h(c)\binom{n}{2} + o(1)\binom{n}{2}}.$$

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The aim of Step 2:

Lemma

For every $c \in (0,1)$ with $c\binom{n}{2}$ being an integer there is $\eta > 0$ s.t.

$$|\text{FORB}_{n,\eta}^*(C_5, c)| \leqslant 2^{(h(c)-r(c))\binom{n}{2}+o(1)\binom{n}{2}},$$

where $r : (0,1) \to \mathbb{R}^+$.

$$\text{FORB}_{n,\eta}^*(C_5, c) = \{G \in \text{FORB}_n^*(C_5, c) : \text{hom}(G) < \eta n\}$$

Regularity Lemma

Szemerédi

Every sufficiently large graph $G = (V, E)$ can be equipartitioned into $V = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_k$ s.t. almost all pairs (V_i, V_j) are ε-regular pairs.

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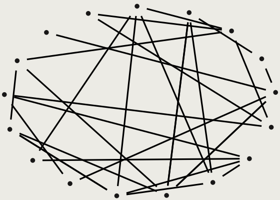

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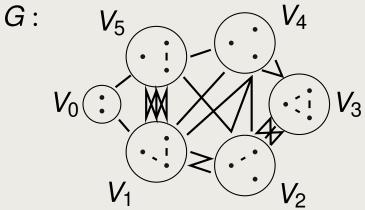

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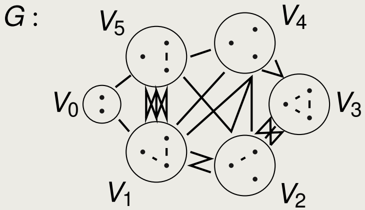

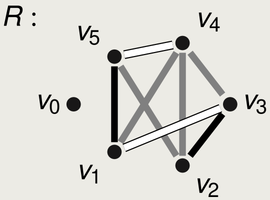

Then G has the coloured reduced graph $R = (\{v_0, \ldots, v_k\}, E_R, \sigma)$ with

$$\sigma : E_R \to \{\blacksquare_{\text{black}}, \blacksquare_{\text{grey}}, \square_{\text{white}}\}, \quad \{v_i, v_j\} \mapsto \begin{cases} \blacksquare_{\text{black}} & \text{if } d(V_i, V_j) > (1-d) \\ \blacksquare_{\text{grey}} & \text{if } d(V_i, V_j) \in [d, 1-d] \\ \square_{\text{white}} & \text{if } d(V_i, V_j) < d. \end{cases}$$

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ε-regular pairs allow the embedding of (fixed) bipartite graphs
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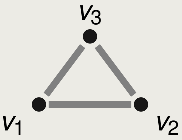
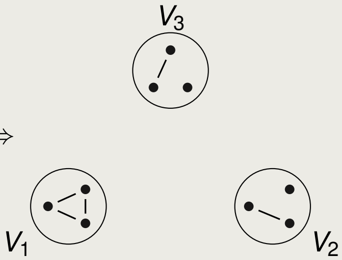


$\Rightarrow$



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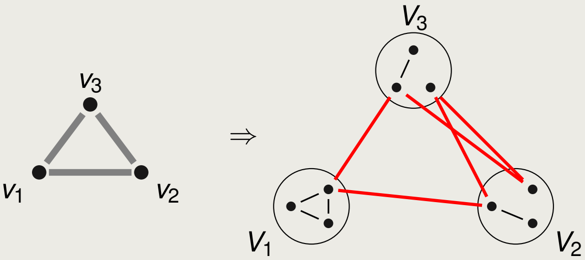

The structure of R

Let $R = (\{v_0, v_1, \dots, v_k\}, E_R)$ be the coloured reduced graph of some $G \in \text{FORB}^*_n(C_5)$. Then at most $k^2/4$ edges in R are grey by Turán's Theorem.

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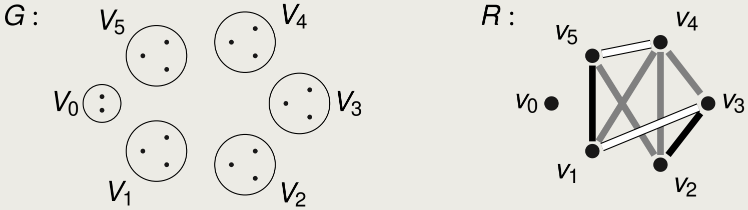

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- edges inside clusters,

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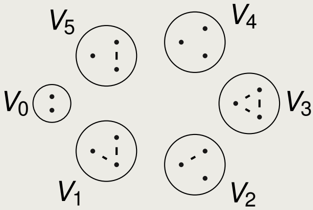

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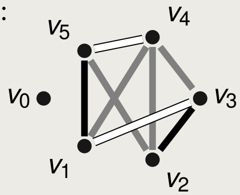

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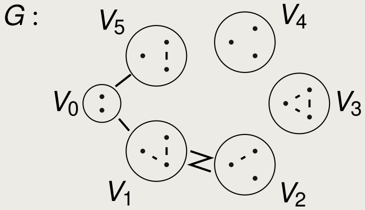

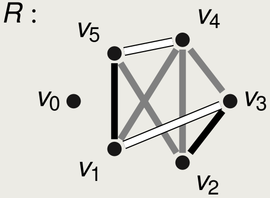

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- edges into white or black edges of R,

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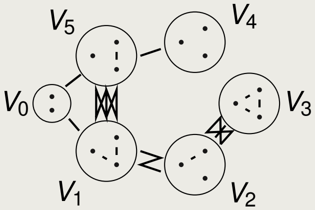

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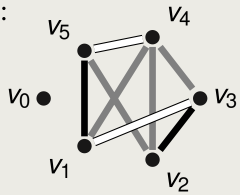

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- edges into grey edges of R.

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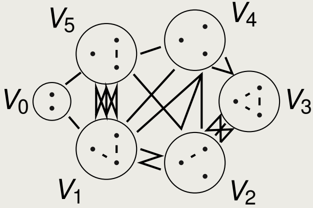

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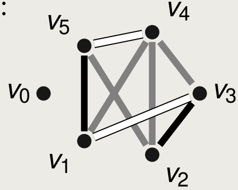

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- edges into white or black edges of R, **negligible**
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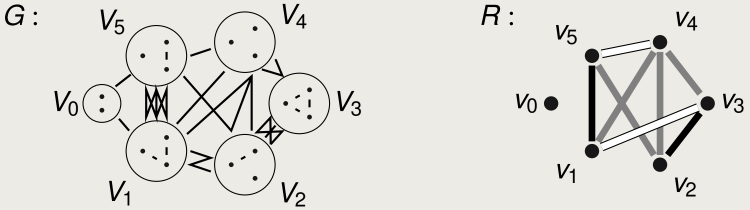

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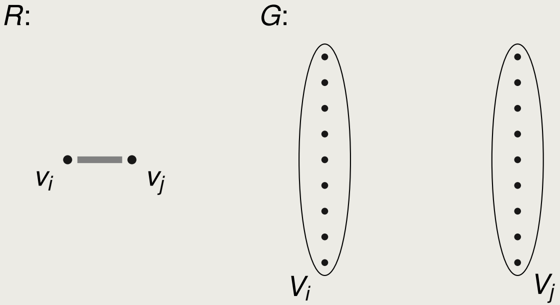

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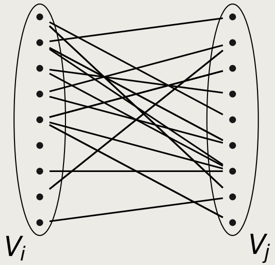

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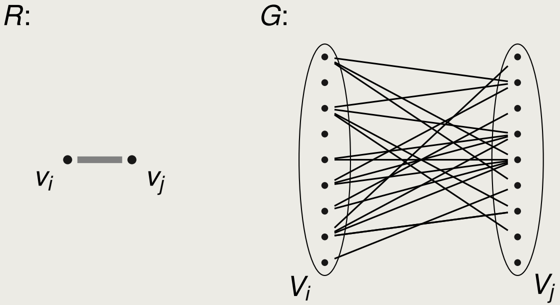
- so we choose at most $c\binom{n}{2}$ out of $n^2/4$ edges

this gives no more than $2^{h(c)\binom{n}{2}+o(n^2)}$ choices



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Let G be a graph on n vertices. Then one of the following is true:

- there is a homogenous set of size $n/6$ in G, or
- G has $n/6$ disjoint copies of P_3, or
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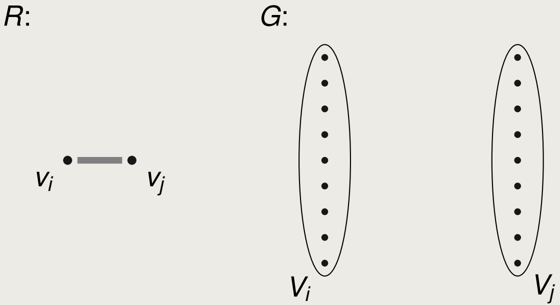

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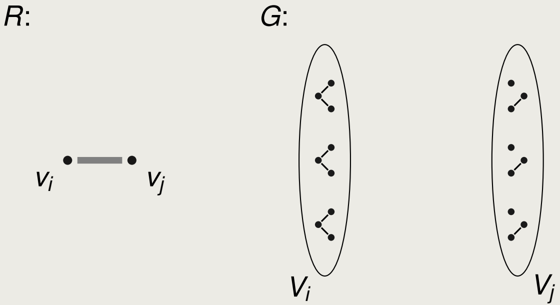

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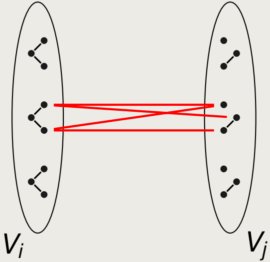

some choices are forbidden!

Lemma

There is $r(c) > 0$ s.t. at most

$$2^{(h(c)-r(c))\binom{n}{2}+o(1)\binom{n}{2}}$$

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- almost all $G \in \textsc{Forb}_n^*(C_5, c)$ have linear homogeneous sets

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almost all $G \in \text{Forb}_n^*(C_5)$ have linear homogeneous sets,
but some not even have polynomial homogeneous sets?