

The Blow-up Lemma and growing degrees

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(joint work with Julia Böttcher, Yoshi Kohayakawa & Anusch Taraz)

- 1 The Blow-up Lemma
 - Its applications
 - And its limits

- 2 Growing degrees
 - Arrangeability of graphs
 - The limits to growth

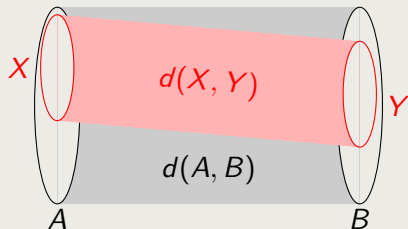
- 3 Sketch of proof
 - Random greedy embedding
 - Auxiliary graphs

Let $\varepsilon, \delta > 0$. The graph $(A \dot{\cup} B, E)$ with $|A| = |B| = n$ is an (ε, δ) -super-regular pair if

- $|d(X, Y) - d(A, B)| \leq \varepsilon$ for all $X \subseteq A, Y \subseteq B$ with $|X|, |Y| \geq \varepsilon n$,
- $\deg(v) \geq \delta n$ for all $v \in A \cup B$.

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- regularity \leftrightarrow
“densities as expected”
- high minimal degree

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The Blow-up Lemma

KOMLÓS, SÁRKÖZY, SZEMERÉDI, '97

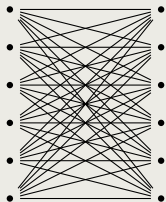
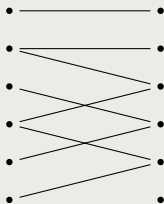
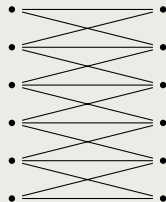
For $\delta > 0, \Delta \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \Delta$ and $H \subseteq K_{n,n}$ then H is subgraph of any (ε, δ) -super-regular pair $(A \dot{\cup} B, E)$ with $|A| = |B| = n$.

another proof was given by:

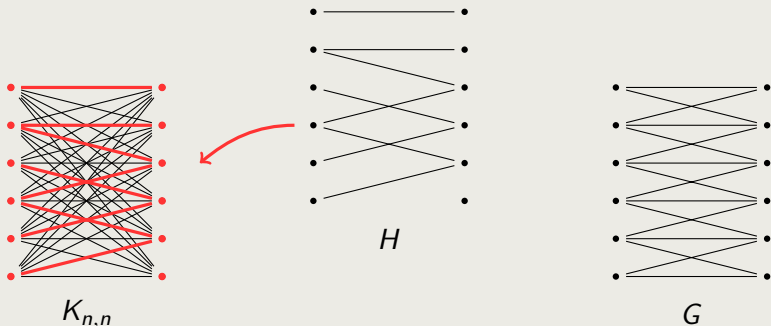
RÖDL, RUCIŃSKI, '99

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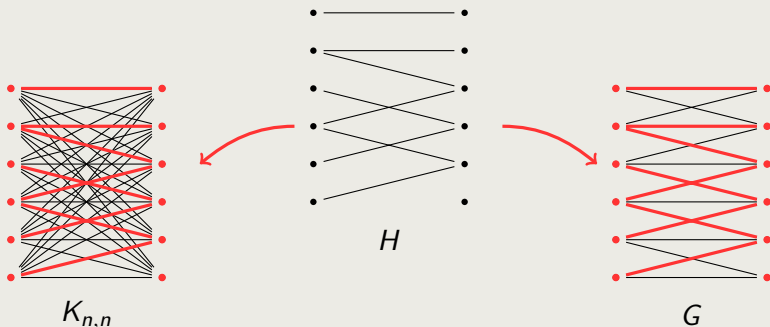

 $K_{n,n}$

 H

 G

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$$H \subseteq K_{n,n}$$

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$$H \subseteq K_{n,n}$$

 \Rightarrow

$$H \subseteq G$$

Applications of the Blow-up Lemma

Theorems

CONJECTURED BY

Graphs with sufficiently high minimal degree contain

- r -th powers of Hamiltonian cycles PÓSA, SEYMOUR, '74
- spanning trees of constant max. degree BOLLOBÁS, '78
- H -factors ALON, YUSTER '96
- spanning planar graphs of constant max. degree BOLLOBÁS, KOMLÓS, '99

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Only spanning subgraphs with **constant maximal degree!**

Spanning subgraphs of growing degrees

Theorem

KOMLÓS, SÁRKÖZY, SZEMERÉDI, '95

Graphs with sufficiently high minimal degree contain

- every tree T of order n with $\Delta(T) \leq cn/\log n$.

Spanning subgraphs of growing degrees

Theorem

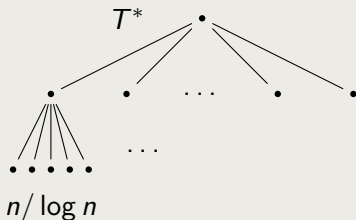
KOMLÓS, SÁRKÖZY, SZEMERÉDI, '95

Graphs with sufficiently high minimal degree contain

- every tree T of order n with $\Delta(T) \leq cn/\log n$.

- best possible because

$$T^* \not\subseteq G_{n,0.9}$$



The Blow-up Lemma for growing degrees

Theorem

KOMLÓS, SÁRKÖZY, SZEMERÉDI, '97

For $\delta > 0$, $\Delta \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \Delta$ and $H \subseteq K_{n,n}$ then H is subgraph of any (ε, δ) -super-regular pair $(A \dot{\cup} B, E)$ with $|A| = |B| = n$.

The Blow-up Lemma for growing degrees

Goal

BÖTTCHER, KOHAYAKAWA, TARAZ, W. '11

For $\delta > 0$, $a \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \sqrt{n}/\log n$ and $H \subseteq K_{n,n}$ is a -arrangeable then H is subgraph of any (ε, δ) -super-regular pair $(A \dot{\cup} B, E)$ with $|A| = |B| = n$.

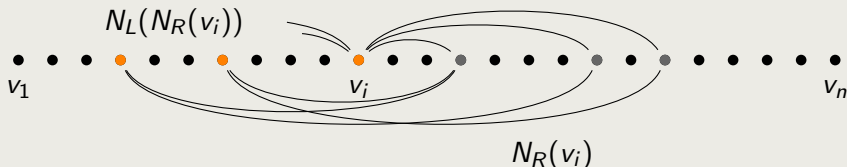
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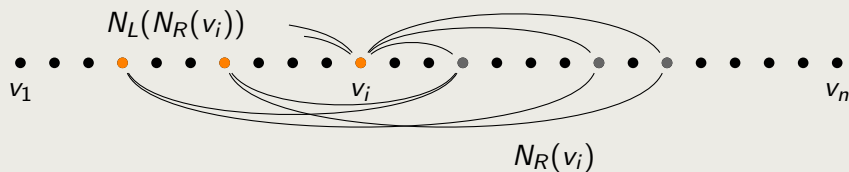
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For $\delta > 0$, $a \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \sqrt{n}/\log n$ and $H \subseteq K_{n,n}$ is a -arrangeable then H is subgraph of any (ε, δ) -super-regular pair $(A \dot{\cup} B, E)$ with $|A| = |B| = n$.

$G = (V, E)$ is a -arrangeable if there is an ordering $V = \{v_1, \dots, v_n\}$ with $|N_L(N_R(v_i))| \leq a$ for all $i = 1, \dots, n$.



Examples of arrangeable graphs



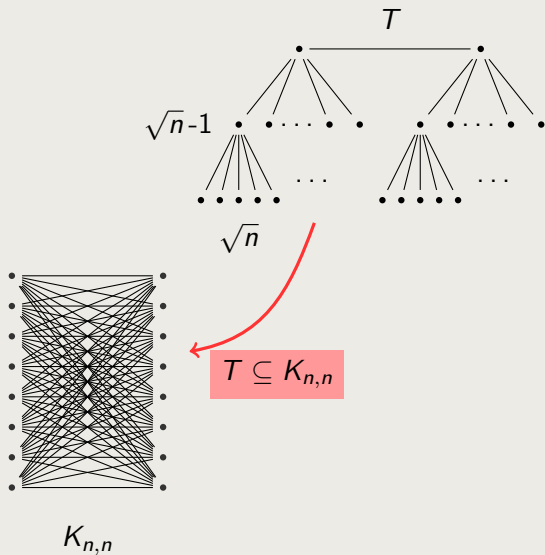
- trees are 1-arrangeable
- planar graphs are 761-arrangeable
- planar graphs are 10-arrangeable
- graphs without a K_p -subdivision are p^8 -arrangeable

CHEN, SHELP '93

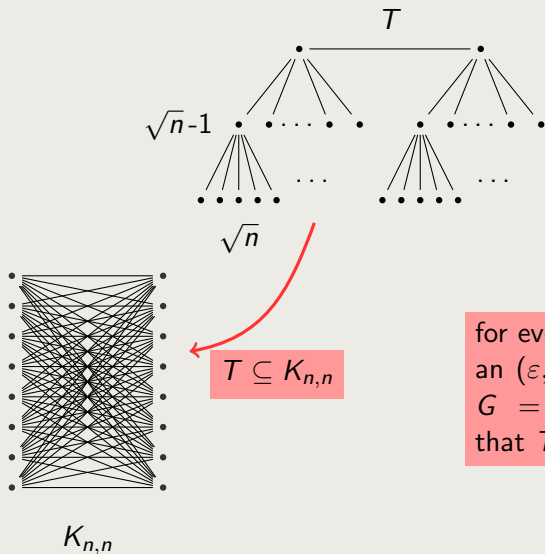
KIERSTEAD, TROTTER '94

RÖDL, THOMAS '94

Why $\Delta(H) \leq \sqrt{n}$?

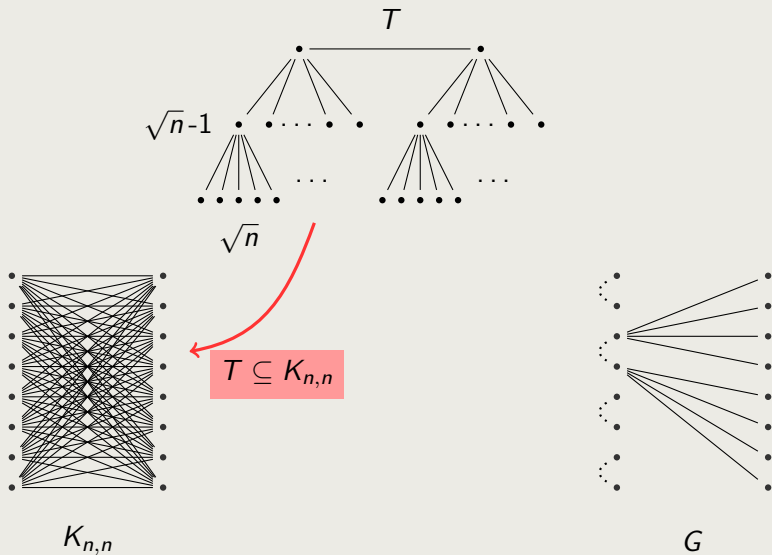


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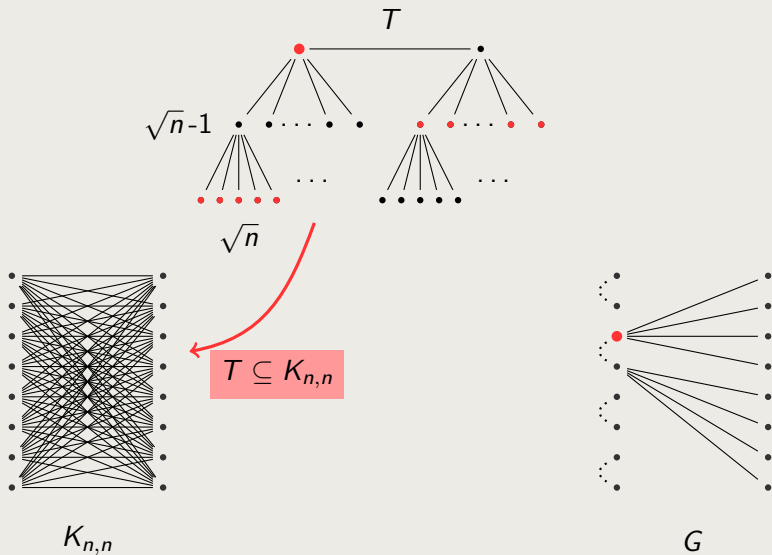


for every $\varepsilon > 0$ there is an $(\varepsilon, 1/2)$ -regular pair $G = (A \dot{\cup} B, E)$ such that $T \not\subseteq G$

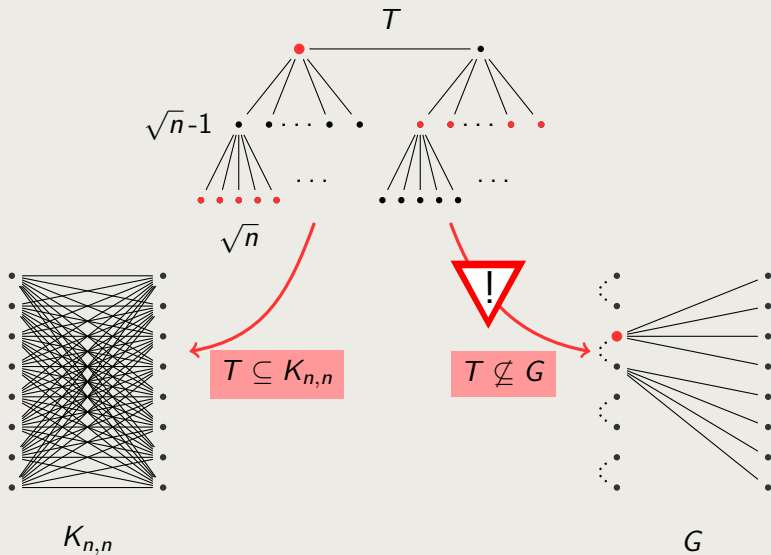
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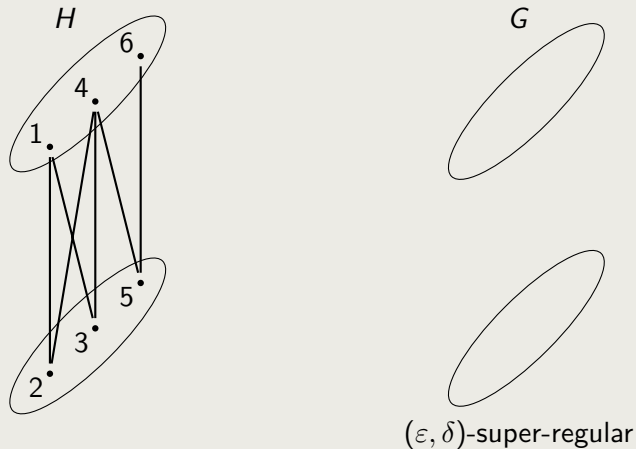
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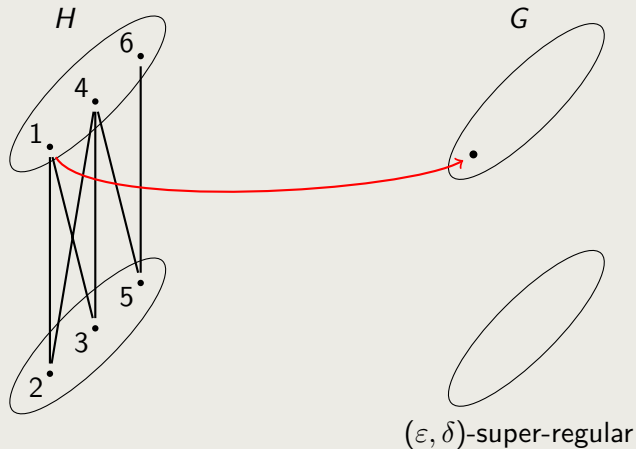
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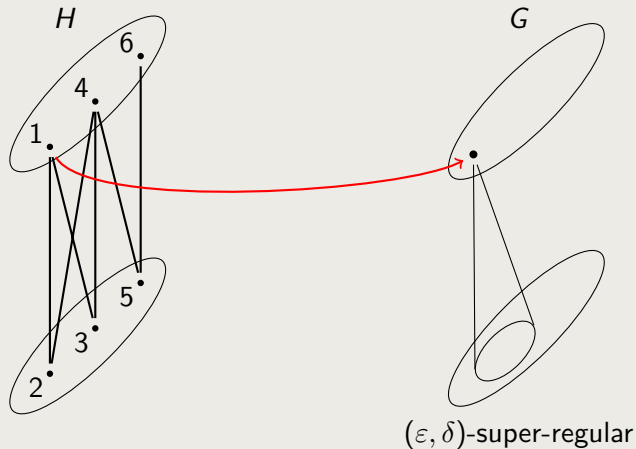
Sketch of proof: a random greedy embedding



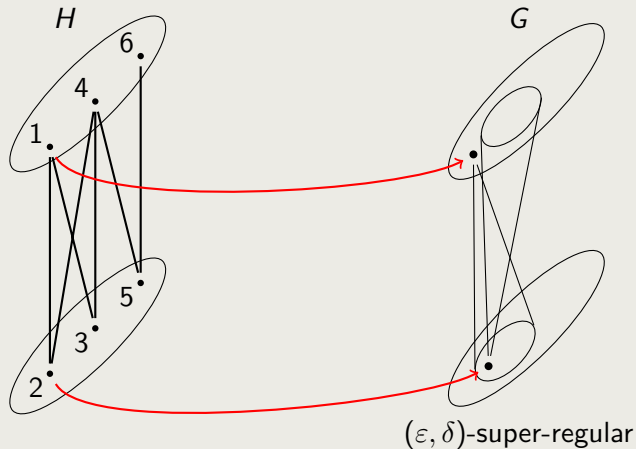
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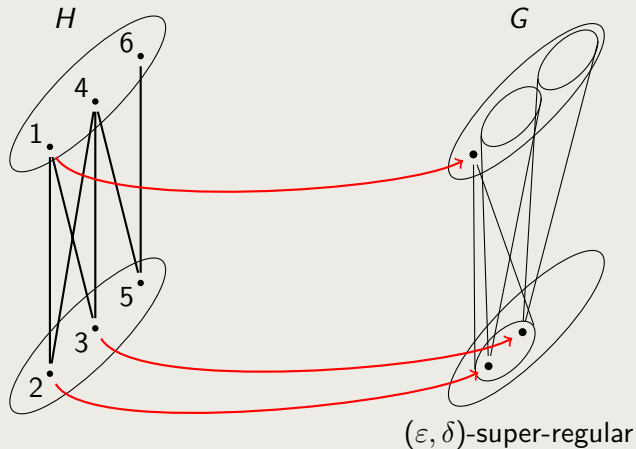
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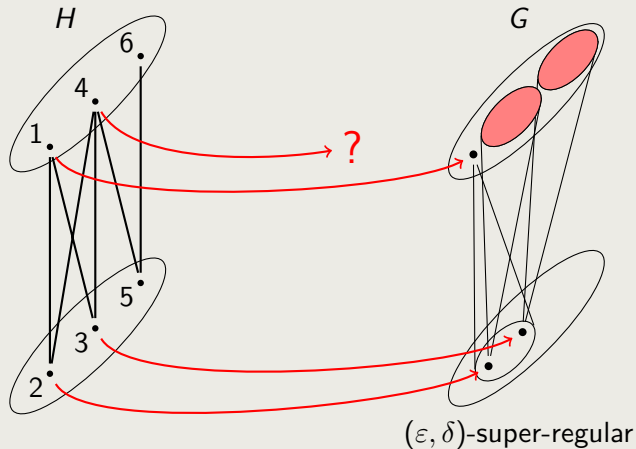
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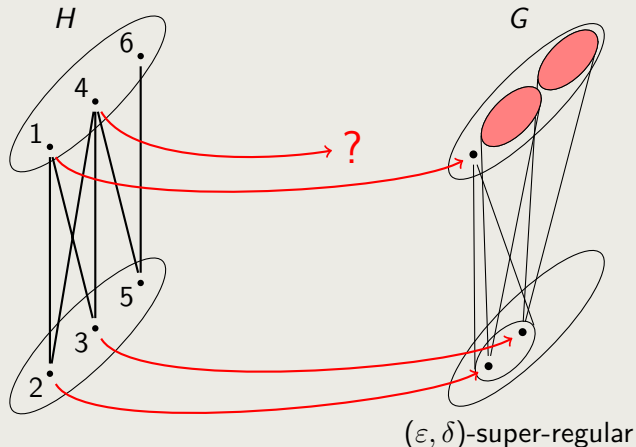
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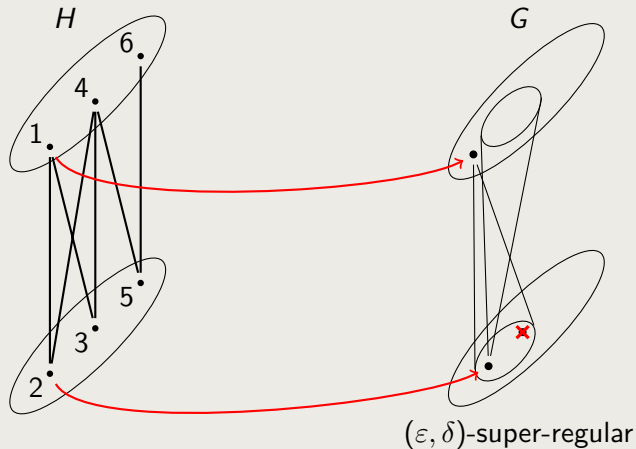


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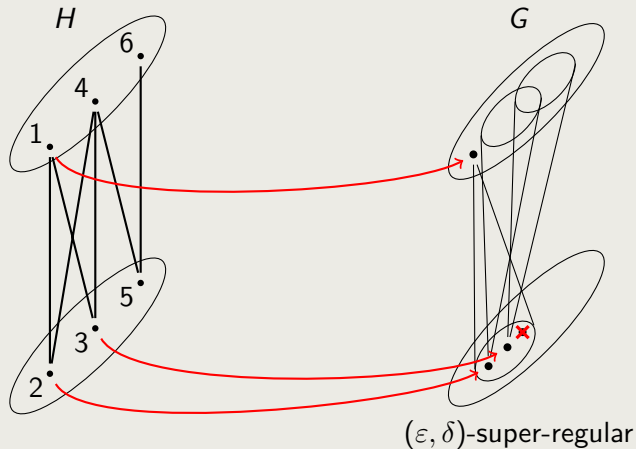


MESSAGE: respect your successors!

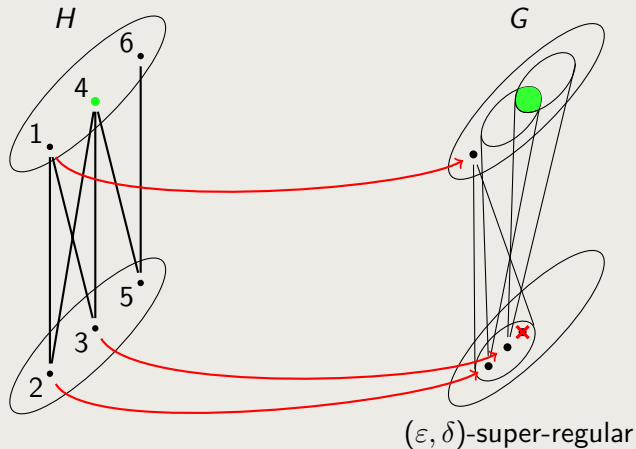
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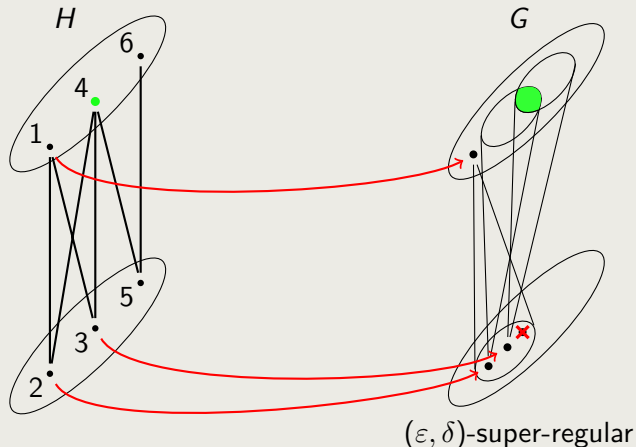
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Does this always work?

Respect your successors. . .

. . . even if their number is growing with n

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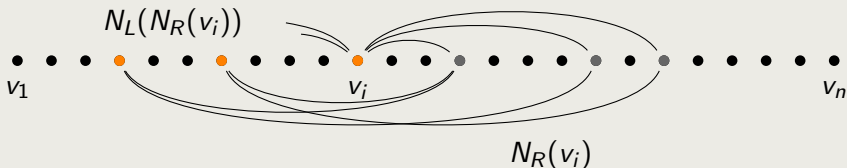
Problem: each successor might “kill” a small linear number of candidates

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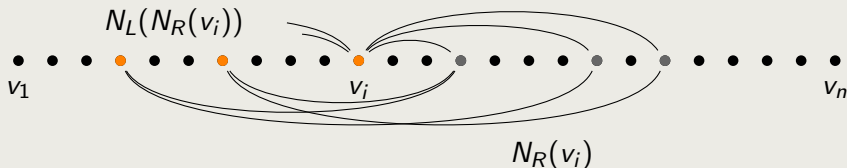


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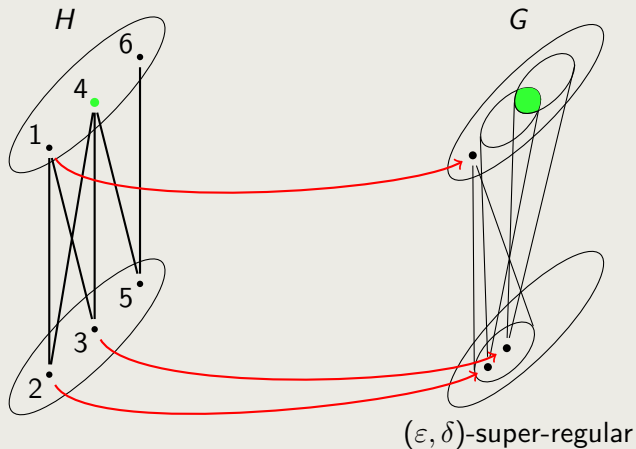
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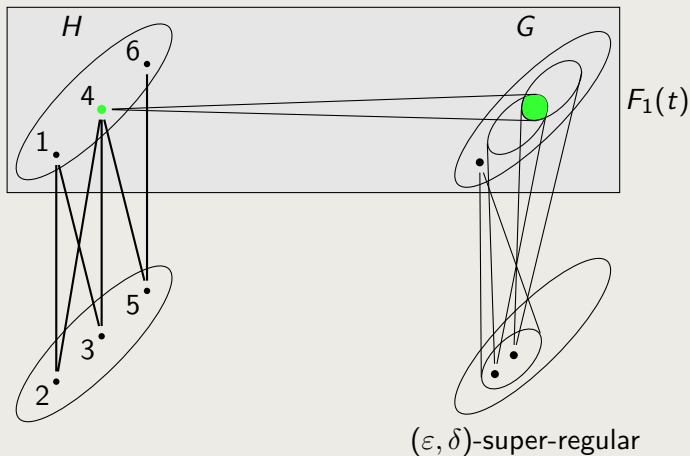
all successors of v_i have at most a predecessors in total

\Rightarrow we have to respect at most 2^a different candidate sets

The auxiliary graphs



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One auxiliary graph, two nice properties

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With probability $1/2$ all auxiliary graphs inherit *some* regularity from G .

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All auxiliary graphs have high minimal degree with probability going to 1.

Therefore:

There is a **perfect matching** in the remaining part of each auxiliary graph that completes the **spanning embedding**.

User's wishlist

- landing zones
- variable cluster sizes
- a sparse version
- what else?