Technische Universität München
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Rotor-Driven Quasirandom Walks

Bachelor Thesis by Carl Georg Heise
– Internet version –

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Declaration

I hereby declare that I have written the Bachelor Thesis on my own and have used no other than the stated sources and aids.

I want to thank Prof. Anusch Taraz for introducing me to the field of quasirandomness and his ongoing help during my studies.

Also, I would like to express my gratitude to my parents Dr. Bettina and Dr. Bodo Heise for their strong support during my time here in Munich and to all people who assisted me during the preparation of this document.

Zusammenfassung

Zufällige Irrfahrten auf Graphen bilden seit langem ein nützliches Werkzeug zur Untersuchung vielerlei natürlicher und abstrakter Phänomene. Im Laufe der Zeit wurden immer wieder Versuche unternommen, diese zufälligen Irrfahrten durch deterministische Prozesse anzunähern. Recht vielversprechend erweist sich dabei das *Rotor-Router* Modell von James Propp, auf welches in dieser Arbeit näher eingegangen wird. Unter anderem werden wir dessen Ähnlichkeit zur klassischen Irrfahrt zeigen, sowie auf Gemeinsamkeiten von aus Irrfahrten hervorgegangenen Modellen eingehen.

Allgemein kann man eine zufällige Irrfahrt folgendermaßen beschreiben: Ein Partikel bewegt sich zufällig auf den Knoten eines Graphen, häufig $\mathbb{Z}^d$, wobei Punkte mit Abstand eins verbunden werden. Eine *quasizufällige* oder auch Rotor-Router genannte Irrfahrt ist im Gegensatz dazu ein Prozess, bei dem das Partikel sich von einem Knoten zum anderen entsprechend einer deterministisch vorgegebenen Rotor-Richtung bewegt.

Wir betrachten quasizufällige Irrfahrten zunächst auf einem sehr allgemeinen Level, beschreiben das Ein-Partikel-Modell und erweitern es anschließend auf mehrere, sich simultan bewegende Partikel. Im Anschluss erläutern wir die Arbeit von Cooper, Doerr, Spencer und Tardos, die gute Schranken für die Unterschiede zwischen quasizufälliger Irrfahrt und dem Erwartungswert der zufälligen Irrfahrt geben. Das hierbei stärkste Resultat begrenzt diese Differenz der Partikelanzahl durch eine Konstante auf einem beliebigen Knoten und gibt weitere, stärkere Abschätzungen für die Gesamtdifferenz auf Intervallen.

Daraufhin gehen wir näher auf die sogenannte *Internal Diffusion Limited Aggregation* (IDLA) ein, bei der die Partikel, anstatt unendlich lange auf $\mathbb{Z}^d$ umherzuwandern, schließlich liegen bleiben und eine Menge *besetzter* Felder erzeugen. Deren Endposition wird dabei anhand des Ergebnisses einer quasi- oder klassisch zufälligen Irrfahrt bestimmt. Alle Simulationen der quasizufälligen IDLA zeigen interessanterweise ausgesprochen kugelförmige Mengen besetzter Felder mit charakteristischen Mustern; ein bisher noch nicht vollständig geklärtes Phänomen.

Levine und Peres waren in der Lage, einige Resultate bezüglich der Kugelförmigkeit zu zeigen. Hierbei bezeichnen wir mit einer guten Kugelförmigkeit eine geringe Differenz der Mittelpunktabstände des entferntesten besetzten Feldes und des nächsten unbesetzten Feldes. Diese Differnz ist durch $\mathcal{O}(r^{(d-1)/d} \log r)$ nach oben beschränkt, wobei r der entsprechende Radius der Kugel ist. Wir betrachten außerdem das als *Divisible Sandpile* oder auch *lineare IDLA* bezeichnete Modell, welches eine noch bessere Kugelförmigkeit erzeugt.

Des Weiteren stellen wir einige mögliche Erweiterungen der quasizufälligen Irrfahrt vor, die eventuell für zukünftige Untersuchungen interessant sein könnten. Unter anderem haben Holroyd und Propp eine Erweiterung auf allgemeine Markovketten vorgeschlagen, welche wir kurz präsentieren. Wir bieten außerdem einen leicht anderen Blickwinkel auf die quasizufällige IDLA, die deren Ähnlichkeit zum Divisible Sandpile besser wiedergibt. Schließlich schlagen wir eine Erweiterung der quasizufälligen IDLA auf allgemeinere zugrundeliegende Gitter als $\mathbb{Z}^d$ vor. Auch diese Berechnungen zeigen fast immer eine außergewöhnlich kugelförmige Struktur. Im Anhang geben wir eine Übersicht über die durchgeführten Berechnungen, zusammen mit Bildern der Rotorpositionen für verschiedene Anzahlen von Partikeln.

Contents

1. Introduction

1.1. Abstract

Random walks on the integers or on any graph have proven to be a very useful tool for studying all kinds of natural and abstract phenomena. Over time, there have been various attempts at approximating this random behavior by a deterministic process. One of the most promising of these is the *rotor-router* model by James Propp, on which this document will focus. Especially, we want to show its similarities to the classical random walk and other models using random walks as underlying process.

On a very basic level, a random walk may be described as a chip moving randomly in-between the vertices of a graph, most commonly $\mathbb{Z}^d$. On the opposite, a *quasirandom* or rotor-router walk is a process where a chip moves from vertex to vertex following given directions represented as rotors on each field. Chapter 2 treats quasirandom walks on a very general level. We first give basic definitions in Section 2.1 and then extend the model to multiple, simultaneously moving chips in Section 2.2, as we are going to see that the one-chip model has extreme cases where the similarity to random walks is not satisfying.

After that, we present the work of Cooper, Doerr, Spencer, and Tardos [CDST07], who have shown strong bounds for the difference between quasirandom walks and the expectation of classical random walks. A constant bound for this difference at an arbitrary point of $\mathbb{Z}^1$ is given in Section 2.3, together with even better estimates for long intervals in space and time. Also, using the so-called *Arrow-forcing Theorem*, one can prove that all bounds given are sharp. Following this, Section 2.4 generalizes the proof for constant bounded differences to $\mathbb{Z}^d$ as done in [CS06] for arbitrary $d \in \mathbb{N}$. This can be seen as the so far strongest result on the similarity of random and quasirandom walks on $\mathbb{Z}^d$.

After studying properties of simple walks, we also present an aggregation model called *internal diffusion limited aggregation* (IDLA) in Section 3.1, where the chips, instead of moving on $\mathbb{Z}^d$ persistently, eventually stop to form a set of *occupied* fields. The position they take is chosen according to an outcome of a random or quasirandom walk. Interestingly, all simulations of the quasirandom IDLA model have shown to produce circular sets of occupied fields with characteristic patterns, a phenomenon that has not yet been fully explained. However, Levine and Peres have some interesting results on the circularity of the aggregation model in [LP09], which will be presented in Section 3.2. If we measure circularity in terms of maximum and minimum distance of the boundary from the origin, their difference is bounded by $\mathcal{O}(r^{(d-1)/d} \log r)$, where r is the radius of the sphere and d the dimension of the model. In Section 3.3, we also present a similar aggregation model, the *divisible sandpile* or *linear*

IDLA, which forms even more accurate balls and has quite a few similarities to quasirandom IDLA.

In Chapter 4 our main purpose is to extend the so far presented models and methods and to give possible directions of research that could be done in the future. First, in Section 4.1, we give a short introduction to general quasirandom Markov chains as defined by Holroyd and Propp in [HP10]. We then offer a different perspective on quasirandom IDLA in Section 4.2, which points out the similarities to the divisible sandpile more clearly and enables a promising way of comparing these two. Finally, we propose an extension of quasirandom IDLA to a more general class of underlying grids than $\mathbb{Z}^d$ in Section 4.3. In most, but not all, of our simulations we could see a similar strong circularity for various grids. An overview of these simulations together with calculations concerning different rotor sequences and higher dimensions is given in Appendix A. We show the final rotor states after a certain number of chips and also the evolution of the radius difference over time, using our own Java implementation of the rotor-router model.

1.2. Notation

In all parts of this thesis, we try to focus on mathematically rigorous statements and proofs. To simplify notation, we are going to use the following conventions and abbreviations.

$\mathbb{R}$	Real numbers
$\mathbb{Z}$	Integers
$\mathbb{N}$	Natural numbers greater than or equal to 1
$\mathbb{N}_{\geq 0}$	Natural numbers greater than or equal to 0
$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$	Ring of congruence classes modulo n
$\mathbf{x}, \mathbf{y} \ldots$	Vectors in $\mathbb{R}^d$ are printed in bold face.
$\mathbf{1}_A$	The function taking value 1 exactly for elements of a set A and 0 otherwise
A^c	The complement of a set A, i. e. the set containing all points not in A
$\#A$	The cardinality of a set A
$d_G(x)$ or $d(x)$	The degree of a vertex x in a graph G. Note that we explicitly allow multiple edges between vertices on a graph. The degree then denotes the number of incident edges.
$N_G(x)$ or $N(x)$	The multiset of all neighboring vertices of a vertex x in a graph G, i. e. we have $\#N_G(x) = d_G(x)$.

2. Quasirandom walks

2.1. Random and quasirandom walks

Definition 2.1.1 (Random Walk)
On an arbitrary countable graph $G = (V, E)$*, let* $X = (X_0, X_1, X_2, \ldots)$ *be a Markov chain on some probability space* $(\Omega, \mathscr{F}, \mathbb{P}_x)$ *with values in* V *and transitional probabilities* $p : V \times V \to [0, 1]$*. Then we call* X *a* (simple) random walk on G *if*

$$p(x, y) = \frac{1}{d(x)} \qquad \text{for all } x, y \in V : xy \in E \text{ and}$$

$$p(x, y) = 0 \qquad \text{otherwise.}$$

We write $\mathbb{P}_x$ *for the probability measure indicating* $X_0 = x$ *and* $\mathbb{E}_x$ *for the expectation with respect to that measure.*

Basically, we place a chip on one vertex of G, which then randomly hops between connected vertices of the graph. In our case, the chip is choosing its next vertex with equal probability among the current vertex' neighbors, though this condition is not absolutely necessary. Random walks in general are a natural choice for simulations, e. g. in biology or finance. However, they also have a lot of nice mathematical properties to study, like hitting times or recurrence, which make them an interesting field of research.

As the main part of this thesis, we want to present a way of simulating a random walk on a graph G using a "quasirandom" deterministic process, which should have similar properties compared to the real random walk. Note that the exact definition of *similarity* here is left open as we might want to focus on different characteristic aspects of random walks.

The rotor model first was described in [PDDK96], which covers steady states of quasirandom walks called "Eulerian walkers". James Propp independently suggested in talks at Berkeley in 2004 [Pro04] to use a similar *rotor-router model*, also called *Propp-machine* after its inventor. It was first formally described by Michael Kleber [Kle05], where he noticed the exceptional similarities between random and quasirandom walks and heuristically found constant bounds for the radius difference of the aggregation model, which we will discuss in Chapter 3.

For the rotor-router model, we install a *rotor* on each vertex of the graph, which can point to any neighbor of the vertex. Similarly to the random walk, we then place a chip on an arbitrary vertex of G. In each step, instead of choosing its next position randomly, the chip will follow the direction to which the rotor on its vertex is pointing. After the step, the used rotor will be rotated to another direction, e.g. clockwise or according to an arbitrary defined sequence of neighbors. Formally, we define the Propp-machine as follows.

Definition 2.1.2 (Quasirandom walk / Propp-machine)
On a countable graph $G = (V, E)$ and for any vertex $x \in V$ let n_x be a bijection between $\mathbb{Z}_{d(x)}$ and the set of its neighbors $N(x) \subseteq V$. Further, let $x_0 \in V$ and $\sigma = \sigma_0 : V \to \mathbb{Z}_{d(.)}$ arbitrary (requiring that $\sigma(x) \in \mathbb{Z}_{d(x)}$ for all $x \in V$).

Then we define the V-valued sequence $X = (X_0, X_1, X_2, \ldots)$ by

$$\begin{aligned} X_0 &= x_0 \\ X_{t+1} &= n_{X_t}\left(\sigma_t(X_t)\right) & \forall t \in \mathbb{N}_{\geq 0}, \end{aligned}$$

where

$$\sigma_{t+1}(x) = \begin{cases} \sigma_t(x) + 1 \pmod{d(x)} & x = X_t \\ \sigma_t(x) & x \neq X_t \end{cases} \qquad \forall t \in \mathbb{N}_{\geq 0}, x \in V.$$

X is called a quasirandom walk *on G, which has position X_t at time $t \in \mathbb{N}_{\geq 0}$.*

The functions n_x are often called the *rotor sequences* as they define in which order the rotor directs to its neighbors. We say σ is the *initial rotor state*, whereas σ_t represents the *rotor state* at time t. Note that for graphs G with multiple edges, the respective rotor directions simply appear more often during one rotation.

Example 2.1.3 (Quasirandom walk on a graph)
This example shows a very simple graph on five vertices, the chip placed on the middle vertex, and the initial rotor state indicated by the black arrows. For a vertex x, we choose n_x such that the rotors always rotate clockwise. After the first step the chip is sent to the rightmost vertex and the central rotor is updated. In each step the chip now moves according to the rotor directions and the state after five steps is shown on the right.

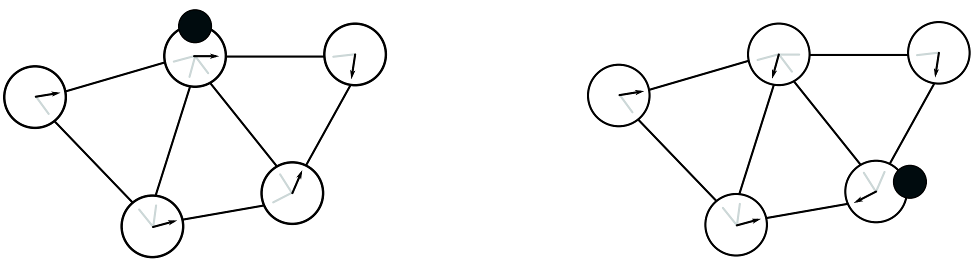


Figure 2.1.: Starting configuration for a quasirandom walk and configuration after five steps.

It is surprising how well such a simple model of a quasirandom walk can simulate a true random walk. As shown in Example 2.1.3, the path the chip is following does not show any obvious pattern and, to an uninformed reader, certainly can look random.

However, one has to note that on a finite graph, as in this example, every quasirandom walk will show periodic behavior eventually. Simply, there are only finitely many configurations for the vector (X_t, σ_t), from which we can follow the existence of $t_1 < t_2$ such that

$$(X_{t_1}, \sigma_{t_1}) = (X_{t_2}, \sigma_{t_2})$$

implying a periodicity of $t_2 - t_1$ or less. In the aforementioned example we have 144 rotor states and 5 chip positions, which means that the periodicity of the quasirandom walk is at most 720.

Therefore, the most interesting questions concerning quasirandom walks arise when looking at infinite graphs. Most commonly we are interested in graphs of the type $G_d = (\mathbb{Z}^d, E)$, where $d \in \mathbb{N}$ and E is such that all points in G_d with Euclidean distance one are pairwise connected. For the sake of simplicity, we will use the terms G_d and $\mathbb{Z}^d$ equivalently in the rest of this thesis unless stated otherwise. Also, we will choose 0 as the initial position for the chip, only neglecting completely equivalent starting configurations.

Example 2.1.4 (Quasirandom walk on $\mathbb{Z}^1$)
Let us look at the graph $\mathbb{Z}^1$ with initial configuration given as in Figure 2.2. We visualize the graph using boxes as this will prove useful in later examples. We start by putting one chip at box 0, following the rules of the quasirandom walk. After moving to boxes 1, 2, . . . , this chip will now wander off to infinity; certainly not a very random behavior.

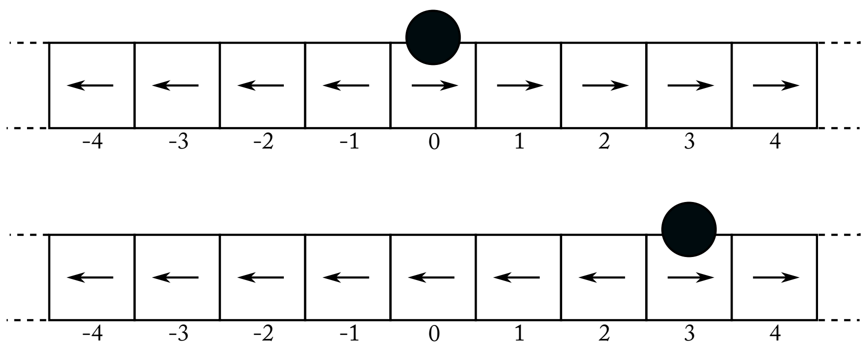


Figure 2.2.: Starting configuration for Example 2.1.4 and configuration after three steps.

So, in an infinite graph, we avoid the periodicity mentioned above, though the level of randomness can be dissatisfactory. We will solve that problem by putting multiple chips on $\mathbb{Z}^d$ at the same time in the next section. The example also showed that the quasirandom walk is not *recurrent*, meaning that there are starting configurations in which the chip will not return to point 0. On the opposite, for random walks it is a well known result from Pólya

[Pól21] that for $d \leq 2$ the probability of the walk returning to 0 equals 1 and is strictly less than 1 for $d > 2$.

Definition 2.1.5 (Hitting time)
Let $X = (X_0, X_1, X_2, \ldots)$ be a random or quasirandom walk on the graph $\mathbb{Z}^d$ for $d \in \mathbb{N}$. Then, following [Kle08, p. 192], we call for any $A \subseteq \mathbb{Z}^d$ the (quasi)random variable

$$\tau_A = \tau_A^X = \inf\{t \geq 0 : X_t \in A\}$$

hitting time *of A. We set $\tau_{\mathbf{x}} = \tau_{\{\mathbf{x}\}}$ for $\mathbf{x} \in \mathbb{Z}^d$.*

Using hitting times, we can show a simple fact random and quasirandom walks have in common. Both of them will leave any bounded regions eventually, though they may return back to the origin infinitely often.

Proposition 2.1.6 (Unboundedness of a random walk on $\mathbb{Z}^d$, [Law96, p. 23])
For any $d \in \mathbb{N}$, a single chip placed on $\mathbf{x} = X_0$ following a random walk on $\mathbb{Z}^d$ will almost surely leave any finite set $S \subset \mathbb{Z}^d$, i. e. $\mathbb{P}_{\mathbf{x}}(\tau_{S^c} < \infty) = 1$.

Proof. Let $S \subset \mathbb{Z}^d$ be finite. We show the more general result that there exist constants $C_S < \infty$ and $\varrho_S < 1$ such that for all $\mathbf{x} \in \mathbb{Z}^d$

$$\mathbb{P}_{\mathbf{x}}(\tau_{S^c} \geq n) \leq C_S \varrho_S^n,$$

as then $\mathbb{P}_{\mathbf{x}}(\tau_{S^c} = \infty) \leq \mathbb{P}_{\mathbf{x}}(\tau_{S^c} \geq n) \leq C_S \varrho_S^n$ for all $n \in \mathbb{N}$. For $\mathbf{x} \notin S$, the result is obvious as $\tau_{S^c} = 0$, so assume otherwise.

Let $R = \sup\{\|\mathbf{x}\|_1 : \mathbf{x} \in S\}$. Then for all $\mathbf{x} \in S$, there exists a path of length $R + 1$ starting at $\mathbf{x}$ and ending outside of S, thus $\mathbb{P}_{\mathbf{x}}(\tau_{S^c} \leq R + 1) \geq (2d)^{-R-1}$. Using the Markov property we get

$$\begin{aligned}\mathbb{P}_{\mathbf{x}}(\tau_{S^c} > k(R+1)) &= \mathbb{P}_{\mathbf{x}}(\tau_{S^c} > (k-1)(R+1)) \cdot \mathbb{P}_{\mathbf{x}}(\tau_{S^c} > k(R+1) | \tau_{S^c} > (k-1)(R+1)) \\ &\leq \mathbb{P}_{\mathbf{x}}(\tau_{S^c} > (k-1)(R+1))(1 - (2d)^{-R-1}) \\ &\leq \varrho_S^{k(R+1)}\end{aligned}$$

for $\varrho_S = (1 - (2d)^{-R-1})^{1/(R+1)}$.

For n arbitrary, split $n = k(R+1) + j$ with $1 \leq j \leq R + 1$. Then

$$\mathbb{P}_{\mathbf{x}}(\tau_{S^c} \geq n) \leq \mathbb{P}_{\mathbf{x}}(\tau_{S^c} > k(R+1)) \leq \varrho_S^{-R-1} \varrho_S^n. \qquad \square$$

Proposition 2.1.7 (Unboundedness of a quasirandom walk on $\mathbb{Z}^d$)
For any $d \in \mathbb{N}$ and for any starting configuration σ, a single chip placed on X_0 following a quasirandom walk on $\mathbb{Z}^d$ will leave any finite set $S \subset \mathbb{Z}^d$, i. e. $\tau_{S^c} < \infty$.

Proof. Without restriction set $X_0 = 0$. Assume there exists such a finite set S where $X_t \in S$ for all $t \in \mathbb{N}_{\geq 0}$ and corresponding initial rotor states σ. As S is finite, there exists at least one $x \in S$ such that $X_t = x$ infinitely often. Let x_S be one of those infinitely often visited x, which is the farthest away from 0. As $X_t = x_S$ infinitely often, also all neighbors of x_S are visited infinitely often, certainly including an x farther away from the origin than x_S and contradicting our assumption. □

2.2. Quasirandom walks with multiple chips

As seen in Example 2.1.4, there are graphs and starting configurations where the quasirandom walk using only one chip is not very interesting. Usually, we are more interested in putting many chips on various positions of $\mathbb{Z}^d$, each of them performing a quasirandom walk. The behavior of many chips now much more resembles what we would expect from a random walk starting from the same chip distribution.

Definition 2.2.1 (Quasirandom walk with multiple chips)
As in the single chip model, for $d \in \mathbb{N}$ and the graph $\mathbb{Z}^d$ let $\sigma_0 : \mathbb{Z}^d \to \mathbb{Z}_{d(.)}$ be an initial rotor state and let $n_{\mathbf{x}}$ be a bijection between $\mathbb{Z}_{d(\mathbf{x})}$ and $N(\mathbf{x}) \subseteq \mathbb{Z}^d$ for any $\mathbf{x} \in \mathbb{Z}^d$. On any vertex $\mathbf{x}$, let us place an arbitrary number of chips as initial distribution.

Now, in each step let all chips perform one single step of the single-chip quasirandom walk, following the rotor direction and updating the rotor before the next chip is moved. We also define $f : \mathbb{Z}^d \times \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ the chip distribution function*, where $f(\mathbf{x}, t)$ is the number of chips on $\mathbf{x}$ at time t and $f(\mathbf{x}, 0)$ the initial number of chips on $\mathbf{x}$.*

Note that f and σ_0 alone define the complete quasirandom walk and that the quasirandom walk is well-defined, as one can easily see that for indiscernible chips it does not matter which chips are moved first if all chips are moved exactly once.

As a small technicality, we say $\mathbf{x} \sim \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ if the (graph) distance $d(\mathbf{x}, \mathbf{y})$ between the vertices is even. A vertex $\mathbf{x} \in \mathbb{Z}^d$ is called *even* if $\mathbf{x} \sim 0$ and an initial distribution $f(\mathbf{x}, 0)$ on $\mathbb{Z}^d$ is called even if all chips are placed on even points. Points and initial distributions are called *odd* respectively. Also, we write $\mathbf{x} \sim t$ for $t \in \mathbb{Z}$ if $d(\mathbf{x}, 0)$ and t have same parity.

The reason is that, $\mathbb{Z}^d$ being a bipartite graph, chips on even positions cannot move to odd positions in an even number of steps and neither can chips on even and odd points interfere directly. However, it is shown in [CDST07] that carefully placed chips on odd vertices can arbitrarily influence the chips starting on even vertices via rotor positions. Therefore, almost all results need to be restricted on even or odd initial distributions.

To compare the random walk with its quasirandom analogon, from now on let $E(\mathbf{x}, t)$ be the *expected* number of chips on $\mathbf{x}$ after performing a random walk with t steps, starting with initial distribution $f(\mathbf{x}, 0)$. Cooper, Doerr, Spencer, and Tardos showed in their work [CDST07, CS06, DF09] that the difference between $E(\mathbf{x}, t)$ and $f(\mathbf{x}, t)$ is much smaller than one would expect at first and in fact bounded by a constant only depending on the dimension d.

The main trick in proving the similarity of quasirandom and random walks is seeing $E(\mathbf{x}, t)$ as a *linear* deterministic random walk where chips are equally distributed to all neighbors, also producing non-integer values of chips. In the next section, we will examine a few of these results, concentrating on the graph $\mathbb{Z}^1$ and discussing higher dimensions in section 2.4.

2.3. Comparing one-dimensional quasirandom and random walks

Definition 2.3.1 (Arrows and Influence, [CDST07])
For a quasirandom walk on $\mathbb{Z}$ with starting distribution $f(x,0)$, let $\mathrm{ARR}(x,t) = (-1)^{\sigma_t(x)}$ be the direction the rotor at $x \in \mathbb{Z}$ is pointing to at time t, also called the arrow direction*, which will replace σ_t in this section.*

Further, we define $E(x, t_1, t_2)$ to be the expected number of chips at x after t_1 quasirandom and $t_2 - t_1$ classical random steps. From that it follows that $E(x,t) = E(x,0,t)$ and $f(x,t) = E(x,t,t)$ for all x and $t \geq 0$. Also, let $H(x,t)$ denote the probability that a chip starting from x and following a random walk is on 0 at time t.

Finally, let $\mathrm{INF}(x,t) = H(x+1,t-1) - H(x,t)$ denote the influence *of a quasirandom step, i. e. the difference of the probabilities of a chip reaching 0, which either performs a random walk for t steps or alternatively is first sent to the right by a single rotor and then following a random walk for only $t-1$ steps.*

Remark. Given the definitions above, the following equations clearly hold for all $x \in \mathbb{Z}$ and $t \in \mathbb{N}_{\geq 0}$.

$$\begin{aligned} \mathrm{ARR}(x,t) &= n_x(\sigma_t(x)) - x \\ \mathrm{ARR}(x,t+1) &= (-1)^{f(x,t)}\,\mathrm{ARR}(x,t) \\ \mathrm{ARR}(x,t+1) &= \mathrm{ARR}(x,t) \text{ for an even initial configuration and } x \not\sim t. \end{aligned} \tag{2.1}$$

Also, in the one-dimensional case we compute

$$H(x,t) = 2^{-t}\binom{t}{(t+x)/2},$$

using the convention that $\binom{t}{s} = 0$ for $s \notin \mathbb{N}$. Out of the t possible steps, the chip needs to take exactly $(t+x)/2$ steps to the right and $(t-x)/2$ steps to the left, all with probability

1/2 for each direction. In addition, we can show that these three representations of INF are equivalent.

$$\begin{aligned}\mathrm{INF}(x,t) &= H(x+1,t-1) - H(x,t) = -\frac{x}{t}H(x,t)\\ &= \frac{1}{2}(H(x+1,t-1) - H(x-1,t-1))\end{aligned} \tag{2.2}$$

The next theorem states that for all (even) initial configurations the difference of the expected number of chips for quasirandom and random walks is bounded by a constant. This is especially interesting as this constant neither depends on the initial number of chips, nor on the time or location the difference is calculated at.

Theorem 2.3.2 (Discrepancy on a single vertex, [CDST07])
For any quasirandom walk on $\mathbb{Z}$ with an even initial configuration, there exists a constant $c_1 \approx 2.29$, independent of the initial distribution $f(x,0)$ such that for all x and t

$$\left| f(x,t) - E(x,t) \right| \leq c_1. \tag{2.3}$$

Proof. Without restriction, we only show $\left| f(0,t) - E(0,t) \right| \leq c_1$ as all initial distributions are equivalent under translation. We will also need the following two lemmas, the first one being an elementary inequality on unimodal functions. A proof for the second, Lemma 2.3.4, can be found in [CDST07].

Lemma 2.3.3
For $X \subseteq \mathbb{R}$, we call $f : X \to \mathbb{R}$ unimodal*, if there is an $m \in X$ such that f is monotonically increasing for $x \leq m$ and monotonically decreasing for $x \geq m$. If $f : X \to \mathbb{R}$ is non-negative and unimodal and if $t_1 < t_2 < \ldots < t_n \in X$ then*

$$\left| \sum_{i=1}^{n} (-1)^i f(t_i) \right| \leq \max_{x \in X} f(x). \tag{2.4}$$

Lemma 2.3.4
For $y \in \mathbb{Z}$, the function $|\mathrm{INF}(y,t)|$ is maximized over all $t \in \mathbb{Z}$ at $t_{\max}(y) = \left\lfloor (y^2 - 4)/3 \right\rfloor + 2$.

We will also use the fact that $\mathrm{INF}(y,t)$ is a unimodal function of t restricted on $t \sim y$ for fixed $y < 0$. We omit the proof for this, as it involves quite a few technical computations. To prove the main theorem, we will first rewrite equation (2.3) as follows.

$$\begin{aligned} f(0,t) - E(0,t) &= E(0,t,t) - E(0,0,t)\\ &= \sum_{s=0}^{t-1} E(0,s+1,t) - E(0,s,t)\end{aligned}$$

Let $0 \leq s < t$ be arbitrary and let ODD_s denote the set of locations, which are occupied by an odd number of chips at time s. Then for all s

$$\begin{aligned} E(0,s+1,t) - E(0,s,t) &= \sum_{y \in \mathrm{ODD}_s} \left(H(y + \mathrm{ARR}(y,s), t-s-1) - H(y, t-s)\right) \\ &= \sum_{y \in \mathrm{ODD}_s} \mathrm{ARR}(y,s)\,\mathrm{INF}(y, t-s). \end{aligned}$$

Let $s_i(y)$ denote the $(i+1)$-th time that y is occupied by an odd number of chips and define $\mathrm{INF}(y,t) = 0$ for $t \leq 0$. Thus, we obtain

$$\begin{aligned} f(0,t) - E(0,t) &= \sum_{s=0}^{t-1} E(0,s+1,t) - E(0,s,t) \\ &= \sum_{s \geq 0} \sum_{y \in \mathrm{ODD}_s} \mathrm{ARR}(y,s)\,\mathrm{INF}(y,t-s) \\ &= \sum_{y \in \mathbb{Z}} \sum_{i \geq 0} \mathrm{ARR}(y, s_i(y))\,\mathrm{INF}(y, t - s_i(y)) \\ &= \sum_{y \in \mathbb{Z}} \mathrm{ARR}(y,0) \sum_{i \geq 0} (-1)^i\,\mathrm{INF}(y, t - s_i(y)) \end{aligned} \tag{2.5}$$

by switching the order of summation and using $\mathrm{ARR}(y, s_i(y)) = -\mathrm{ARR}(y, s_{i-1}(y))$ from the definition of s_i. Now applying Lemmas 2.3.3 and 2.3.4 to the unimodal function $\mathrm{INF}(y,t)$ finally yields

$$\begin{aligned} \left|f(0,t) - E(0,t)\right| &\leq \sum_{y \in \mathbb{Z}} \left| \sum_{i \geq 0} (-1)^i\,\mathrm{INF}(y, t - s_i(y)) \right| \\ &\leq \sum_{y \in \mathbb{Z}} \max_{t \in \mathbb{Z}} \left|\mathrm{INF}(y,t)\right| \\ &= 2 \sum_{y=1}^{\infty} \left|\mathrm{INF}(y, t_{\max}(y))\right| \\ &= c_1 \approx 2.29, \end{aligned}$$

which gives the constant bound. □

The main idea of this proof basically is the *hybrid* process created using the mixed expectations $E(x, t_1, t_2)$. We can transform the quasirandom walk to a random walk by stopping the quasirandom behavior after a certain number of steps, a trick that will be used on quite a few other occasions. Note that Example 2.1.4 does not violate this theorem, as $\left|f(x,t) - E(x,t)\right| < 1 < 2.29$ at all times for that particular initial configuration. However, this also shows that a constant as large as 2.29 will not be of any use in analyzing the quasirandom behavior of single chips.

The next result shows that we can force the positions of the arrows on $\mathbb{Z}^1$ at all times by using a tailored initial distribution. This will become useful in proving c_1 to be a sharp upper bound for $|f(x,t) - E(x,t)|$.

Theorem 2.3.5 (Arrow-forcing Theorem, [CDST07])
Let $\varrho(x,t) : \mathbb{Z}^1 \times \mathbb{N}_{\geq 0} \to \{-1,+1\}$ be arbitrarily defined for all $x \sim t$. Then there exists an even initial configuration for a quasirandom walk that results in $\mathrm{ARR}(x,t) = \varrho(x,t)$ *for all such x and t. Similarly, if $\varrho(x,t)$ is defined for $x \sim t+1$ a suitable odd configuration can be found.*

Proof. We will only prove the statement for even initial distributions. First, let $f_0(x,0) = 0$ for all x be the empty distribution and let $\mathrm{ARR}_0(x,0) = \varrho(x,0)$ for even x and $\mathrm{ARR}_0(x,1) = \varrho(x,1)$ for odd x the starting arrow positions. For a quasirandom walk starting with f_0, we now have $\mathrm{ARR}(x,t) = \varrho(x,t)$ for $0 \leq t \leq 1$ and $x \sim t$.

Assume now that, for some $T \geq 0$, we have $\mathrm{ARR}_T(x,t) = \varrho(x,t)$ for $0 \leq t \leq T+1$ and $x \sim t$, using an initial distribution $f_T(x,0)$. We then define $f_{T+1}(x,0) = f_T(x,0) + \varepsilon_x 2^T$ for even x, $f_{T+1}(x,0) = 0$ for odd x, and for $\varepsilon_x \in \{0,1\}$ to be specified.

We want to choose ε_x such that, using the new initial distribution, $\mathrm{ARR}_{T+1}(x,t) = \varrho(x,t)$ holds for $0 \leq t \leq T+2$ and $x \sim t$. For $t \leq T$, this is obvious, as the additional pile of 2^T chips will split T times without changing rotor directions. In the case of $x \sim T+1$ we get $\mathrm{ARR}_{T+1}(x,T+1) = \mathrm{ARR}_T(x,T) = \mathrm{ARR}_T(x,T+1) = \varrho(x,T+1)$, using (2.1), because the initial distribution was even. For $x \sim T+2$, note that

$$\mathrm{ARR}_{T+1}(x,T+2) = (-1)^{f_{T+1}(x,T)}\,\mathrm{ARR}_{T+1}(x,T) = (-1)^{f_{T+1}(x,T)}\varrho(x,T).$$

To guarantee $\mathrm{ARR}_{T+1}(x,T+2) = \varrho(x,T+2)$, we need $f_{T+1}(x,T)$ to be even if and only if $\varrho(x,T+2) = \varrho(x,T)$. By definition of f_{T+1} we have

$$f_{T+1}(x,T) = f_T(x,T) + \sum_{y \text{ even}} \varepsilon_y \binom{T}{(T+x-y)/2}. \tag{2.6}$$

We now need to define $\varepsilon_x \in \{0,1\}$, so that the parity of $f_{T+1}(x,T)$ is as needed. For $T = 0$, we have $f_{T+1}(x,T) = f_T(x,T) + \varepsilon_x$, which is trivial. For $t > 0$, note that the elements of the sum above are almost always zero and we split

$$\sum_{y \text{ even}} \varepsilon_y \binom{T}{(T+x-y)/2} = \varepsilon_{x+T} + h_x + \varepsilon_{x-T}$$

where h_x only depends on ε_y with $x - T < y < x + T$. Now calculate the ε_y iteratively, initializing $\varepsilon_y = 0$ for $-T < y \leq T$ and setting the other ε_y stepwise to ensure the correct parity of $f_{T+1}(y-T,T)$ and $f_{T+1}(y+T,T)$ for $y > T$ and $y \leq T$ respectively.

Using this procedure, one can generate a sequence $f_0(x,0), f_1(x,0), \ldots$ of initial distributions, converging to an initial distribution $f(x,0)$, as $f_t(x,0) = f_T(x,0)$ for all $t \geq T$ and $|x| < T$. This initial distribution satisfies the statement of the theorem. □

This theorem is a nice example of how one can tweak the initial configuration of a quasi-random walk to achieve certain specified rotor positions or distributions at any given time and place. Let $y > 0$ arbitrary, $t_0 = t_{\max}(y)$ (c.f. Lemma 2.3.4), and

$$\varrho(x,t) = \begin{cases} -1 & \text{if } x > 0 \text{ and } t \le t_0 - t_{\max}(x) \\ -1 & \text{if } x < 0 \text{ and } t > t_0 - t_{\max}(x) \\ 1 & \text{otherwise.} \end{cases}$$

The Arrow-forcing Theorem now guarantees an initial distribution such that $\mathrm{ARR}(x,t) = \varrho(x,t)$ for all $x \sim t$. One can show that for any $|x| \le y, x \ne 0$, there is an odd number of chips on x exactly at time $t_0 - t_{\max}(x)$, which, using (2.5), leads to

$$\left|f(0,t_0) - E(0,t_0)\right| = 2\sum_{x=1}^{y} \left|\mathrm{INF}(x, t_{\max}(x))\right| \xrightarrow{y\to\infty} c_1.$$

This shows that c_1 is actually a sharp upper bound for $\left|f(x,t) - E(x,t)\right|$. Although a maximum constant difference between expected random walks and quasirandom walks is quite nice, we can further improve this result if looking at differences between these two on intervals in space and time.

Theorem 2.3.6 (Discrepancy on intervals in space, [CDST07])
For any quasirandom walk on $\mathbb{Z}^1$ with an even initial configuration, for all $t \ge 0$, and for all intervals $X \subseteq \mathbb{Z}$ of length L, we have

$$\left|\sum_{x\in X} f(x,t) - E(x,t)\right| = \mathcal{O}(\log L).$$

Also, for every $L > 0$, there exist an initial configuration, $t \ge 0$ and an interval X of length L such that

$$\left|\sum_{x\in X} f(x,t) - E(x,t)\right| = \Omega(\log L).$$

Proof. We will only give a short sketch of the proof. Without restriction we can assume $X = \{-L+1, -L+2, \dots, 0\}$. Using the same method as in (2.5), we can show for all x

$$f(x,t) - E(x,t) = \sum_{y\in\mathbb{Z}} \mathrm{ARR}(y,0) \sum_{i\ge 0} (-1)^i \,\mathrm{INF}(y-x, t-s_i(y)) \tag{2.7}$$

and thus have

$$\begin{aligned}\sum_{x\in X} f(x,t) - E(x,t) &= \sum_{y\in\mathbb{Z}} \mathrm{ARR}(y,0) \sum_{x\in X}\sum_{i\ge 0} (-1)^i \,\mathrm{INF}(y-x, t-s_i(y)) \\ &= \sum_{y\in\mathbb{Z}} \mathrm{CON}(y)\end{aligned}$$

where we define the contribution of y by

$$\mathrm{CON}(y) = \mathrm{ARR}(y,0)\sum_{x\in X}\sum_{i\geq 0}(-1)^i\,\mathrm{INF}(y-x,t-s_i(y)).$$

Again, using Lemmas 2.3.3 and 2.3.4, one can show that for all y with distance to X at least L

$$|\,\mathrm{CON}(y)| = \mathcal{O}(Ly^{-2}),$$

leading to

$$\sum_{y>L}|\,\mathrm{CON}(y)| = \mathcal{O}(1) \text{ and } \sum_{y\leq -2L}|\,\mathrm{CON}(y)| = \mathcal{O}(1). \tag{2.8}$$

For all other y, define $\delta_1 = \mathbf{1}_{\{t\sim L\}}$ and $\delta_2 = \mathbf{1}_{\{t\sim 0\}}$. We then use that $t \mapsto H(x,t)$ is unimodal for all $x \sim t$, thus by Lemma 2.3.3 and (2.2)

$$\begin{aligned}
|\,\mathrm{CON}(y)| &= \left|\sum_{i\geq 0}(-1)^i\sum_{x\in X}\mathrm{INF}(y-x,t-s_i(y))\right| \\
&= \frac{1}{2}\left|\sum_{i\geq 0}(-1)^i\sum_{x\in X}\big(H(y-x+1,t-s_i(y)-1)-H(y-x-1,t-s_i(y)-1)\big)\right| \\
&= \frac{1}{2}\left|\sum_{i\geq 0}(-1)^i\,\big(H(y+L-\delta_1,t-s_i(y)-1)-H(y-\delta_2,t-s_i(y)-1)\big)\right| \\
&\leq \frac{1}{2}\max_{s\in\mathbb{N}_{\geq 0}}\big|H(y+L-\delta_1,s)\big| + \frac{1}{2}\max_{s\in\mathbb{N}_{\geq 0}}\big|H(y-\delta_2,s)\big| \\
&= \mathcal{O}\left((y+L-1/2)^{-1}\right) + \mathcal{O}\left((y-1/2)^{-1}\right).
\end{aligned}$$

Together with (2.8), this sums up to

$$\begin{aligned}
\left|\sum_{x\in X}f(x,t)-E(x,t)\right| &\leq \sum_{y\leq -2L}|\,\mathrm{CON}(y)| + \sum_{y=-2L+1}^{L}|\,\mathrm{CON}(y)| + \sum_{y>L}|\,\mathrm{CON}(y)| \\
&= \mathcal{O}(1) + \sum_{y=-2L+1}^{L}\mathcal{O}\left((y+L-1/2)^{-1}\right) + \mathcal{O}\left((y-1/2)^{-1}\right) \\
&= \mathcal{O}(\log L),
\end{aligned}$$

which is what we wanted to prove. The lower bound in the second part can be shown similarly to the single vertex discrepancy using the Arrow-forcing Theorem. □

We have shown that, although the single vertex discrepancy is only bounded by a constant, over long intervals in space this bound will not be sharp for almost all vertices. A similar statement shows that this is equally true for intervals in time, though the proof does not give many new insights and will be omitted.

Theorem 2.3.7 (Discrepancy on intervals in time, [CDST07])
For any quasirandom walk on $\mathbb{Z}^1$ with an even initial configuration, for all $x \in \mathbb{Z}$, and for all intervals $S \subseteq \mathbb{N}_{\geq 0}$ of length T, we have that

$$\left| \sum_{s \in S} f(x,s) - E(x,s) \right| = \mathcal{O}(\sqrt{T})$$

and that this bound is sharp.

2.4. Comparing multidimensional quasirandom and random walks

The last section gave an overview of the results which are known for the one-dimensional quasirandom walk. Intuitively, most statements for the discrepancy between quasirandom and random walks should remain true for higher dimensions d. However, they turn out to be slightly more technical to prove, e. g. the higher number of neighbors per vertex results in various possible sequences of rotor directions instead of the simple “right–left”.

Definition 2.4.1
As before, let $H(\mathbf{x}, t)$ denote the probability that a chip starting from $\mathbf{x} \in \mathbb{Z}^d$ and following a random walk is on 0 at time t. Also, for $j \in \mathbb{Z}_{2d}$, let $\mathrm{INF}^j(\mathbf{x}, t) = H(\mathbf{x}, t) - H(\mathbf{x} - \mathbf{e}_j, t-1)$ denote the influence *of a quasirandom step into direction j where $\mathbf{e}_j = n_{\mathbf{x}}(j)$ denotes the j-th principal direction at vertex $\mathbf{x}$.*

Remark. One can easily show that for all t

$$\sum_{j \in \mathbb{Z}_{2d}} \mathrm{INF}^j(\mathbf{x}, t) = 0. \tag{2.9}$$

Theorem 2.4.2 (Discrepancy on a single vertex, [CS06])
For any $d \in \mathbb{N}$ and for any quasirandom walk on $\mathbb{Z}^d$ with an even initial configuration, there exists a constant c_d, independent of the initial distribution $f(\mathbf{x}, 0)$, such that for all $\mathbf{x}$ and t

$$\left| f(\mathbf{x}, t) - E(\mathbf{x}, t) \right| \leq c_d.$$

Proof. Without restriction, set $\mathbf{x} = 0$ and fix t. For any $\mathbf{y} \in \mathbb{Z}^d$, let $M_{\mathbf{y}}$ be the total number of times a chip was moved from $\mathbf{y}$ and $t_0 \leq \ldots \leq t_i \leq \ldots \leq t_{M_{\mathbf{y}}-1} \leq t$ the time when the i-th chip was moved. We define

$$\Delta_{\mathbf{y}} = \sum_{i=0}^{M_{\mathbf{y}}-1} \mathrm{INF}^{\sigma_0(\mathbf{y})+i}(\mathbf{y}, t - t_i)$$

the *total contribution* of $\mathbf{y}$ to $f(0,t) - E(0,t)$. Similar to (2.5) and using the hybrid process $E(0, t_1, t_2)$, we split and rearrange sums into

$$f(0,t) - E(0,t) = \sum_{\mathbf{y}\in\mathbb{Z}^d} \Delta_{\mathbf{y}} = \sum_{\mathbf{y}\in\mathbb{Z}^d} \sum_{i=0}^{M_{\mathbf{y}}-1} \mathrm{INF}^{\sigma_0(\mathbf{y})+i}(\mathbf{y}, t - t_i). \tag{2.10}$$

From now on, we will assume (and need) $d \geq 2$, the case $d = 1$ having already been treated in Theorem 2.3.2. It is obviously sufficient to show the following claim, as then

$$|f(0,t) - E(0,t)| \leq \sum_{\mathbf{y}\in\mathbb{Z}^d} |\Delta_{\mathbf{y}}| = \mathcal{O}\left(\sum_{\mathbf{y}\in\mathbb{Z}^d} \|\mathbf{y}\|^{-d-1} \log^{d-1} \|\mathbf{y}\| \right) = \mathcal{O}(1).$$

Claim. Let $\|\cdot\|$ be the 2-norm on $\mathbb{R}^d$. Then $|\Delta_{\mathbf{y}}| = \mathcal{O}\left(\|\mathbf{y}\|^{-d-1} \log^{d-1} \|\mathbf{y}\|\right)$ as $\|\mathbf{y}\| \to \infty$ and $|\Delta_{\mathbf{y}}| < K_{d,\mathbf{y}}$ for some constant $K_{d,\mathbf{y}}$, both independent of $t_0, \ldots, t_{M_{\mathbf{y}}-1}$, σ_0, and the initial distribution $f(\mathbf{x}, 0)$.

Let $\varphi = \sigma_0(\mathbf{y})$. Applying the definition of $\Delta_{\mathbf{y}}$ and (2.9) yields

$$\Delta_{\mathbf{y}} = \sum_{\substack{i=0\\ \varphi+i\neq 0}}^{M_{\mathbf{y}}-1} \mathrm{INF}^{\varphi+i}(\mathbf{y}, t - t_i) - \sum_{\substack{i=0\\ \varphi+i=0}}^{M_{\mathbf{y}}-1} \sum_{0\neq j\in\mathbb{Z}_{2d}} \mathrm{INF}^{j}(\mathbf{y}, t - t_i)$$

and after rearranging sums

$$= \sum_{0\neq j\in\mathbb{Z}_{2d}} \left(\sum_{\substack{i=0\\ \varphi+i=j}}^{M_{\mathbf{y}}-1} \mathrm{INF}^{j}(\mathbf{y}, t - t_i) - \sum_{\substack{i=0\\ \varphi+i=0}}^{M_{\mathbf{y}}-1} \mathrm{INF}^{j}(\mathbf{y}, t - t_i) \right)$$

$$= \sum_{0\neq j\in\mathbb{Z}_{2d}} (\pm 1) \cdot \sum_{i=0}^{s_j} (-1)^i \, \mathrm{INF}^{j}(\mathbf{y}, t - t'_i)$$

where $t'_0, \ldots, t'_{s_j}$ for $j \in \mathbb{Z}_{2d}$ is a corresponding subsequence of $t_0, \ldots, t_{M_{\mathbf{y}}-1}$. Without restriction we can assume $t'_0 < \ldots < t'_{s_j}$, as equal entries within the sum would cancel each other out.

It therefore suffices to show that for all j and s and for some sequence $t_0 < \ldots < t_s$

$$\sum_{i=0}^{s} (-1)^i \, \mathrm{INF}^{j}(\mathbf{y}, t - t_i) = \mathcal{O}\left(\|\mathbf{y}\|^{-d-1} \log^{d-1} \|\mathbf{y}\|\right)$$

and that this value is bounded by a constant for any fixed $\mathbf{y}$. By Definition 2.4.1 of influence, we have

$$\sum_{i=0}^{s}(-1)^i \operatorname{INF}^j(\mathbf{y}, t-t_i) = \sum_{i=0}^{s}(-1)^i \left(H(\mathbf{x}, t-t_i) - H(\mathbf{x}-\mathbf{e}_j, t-t_i-1)\right)$$
$$= \sum_{i=0}^{s}\frac{1}{2d}\sum_{k\in\mathbb{Z}_{2d}}(-1)^i \left(H(\mathbf{x}-\mathbf{e}_k, t-t_i-1) - H(\mathbf{x}-\mathbf{e}_j, t-t_i-1)\right).$$

For fixed j,k, let $\mathbf{z}=\mathbf{y}-\mathbf{e}_j$, $\mathbf{e}'=\mathbf{e}_j-\mathbf{e}_k$, and $\nabla_{\mathbf{e}'}(\mathbf{z},t)=H(\mathbf{x}+\mathbf{e}',t)-H(\mathbf{x},t)$. After switching sums, this gives

$$\sum_{i=0}^{s}(-1)^i \operatorname{INF}^j(\mathbf{y}, t-t_i) = \frac{1}{2d}\sum_{k\in\mathbb{Z}_{2d}}\sum_{i=0}^{s}(-1)^i \nabla_{\mathbf{e}'}(\mathbf{z}, t-t_i).$$

As $\mathbf{y}$ and $\mathbf{z}$ are asymptotically equal we have therefore reduced the claim to show that for arbitrary $\mathbf{e}'=\mathbf{e}_j-\mathbf{e}_k$, s, and $t_0<\ldots<t_s$

$$\sum_{i=0}^{s}(-1)^i \nabla_{\mathbf{e}'}(\mathbf{z}, t-t_i) = \mathcal{O}\left(\|\mathbf{z}\|^{-d-1}\log^{d-1}\|\mathbf{z}\|\right) \tag{2.11}$$

and that this sum is also bounded by a constant for fixed $\mathbf{z}$.

From [Law96] we know that there exists a constant $c>0$ such that a random walk on $\mathbb{Z}^d$ of length t and starting at the origin ends at a point at least $\alpha\sqrt{t}$ away with probability lower than $ce^{-\alpha}$. By using this as a simple upper bound, we have $|\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)| \le 2ce^{(-\|\mathbf{z}\|+\|\mathbf{e}'\|)/\sqrt{t-t_i}}$. For t_i close to t, this means that $|\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)|$ is actually very small: Let $T=t-\frac{\|\mathbf{z}\|^2}{(d+3)^2\log^2\|\mathbf{z}\|}$. Then, for all $t\ge t_i\ge T$, we obtain $|\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)| = \mathcal{O}\left(\|\mathbf{z}\|^{-d-3}\right)$ and by summing up

$$\sum_{\substack{i=0\\ t_i\ge T}}^{s}|\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)| \le \|\mathbf{z}\|^2\mathcal{O}\left(\|\mathbf{z}\|^{-d-3}\right) = \mathcal{O}\left(\|\mathbf{z}\|^{-d-1}\right). \tag{2.12}$$

Define $\vartheta_i = t-t_i$ and $p(\mathbf{z},\vartheta) = 2\left(\frac{d}{2\pi\vartheta}\right)^{d/2} e^{-d\|\mathbf{z}\|^2/2\vartheta}$. The *Local Central Limit Theorem* in [Law96, p. 14] yields for $\mathbf{z}\sim\vartheta_i$

$$\left|\nabla_{\mathbf{e}'}(\mathbf{z},\vartheta_i) - p(\mathbf{z}+\mathbf{e}',\vartheta_i) + p(\mathbf{z},\vartheta_i)\right| = \|\mathbf{z}\|^{-2}\mathcal{O}\left(\vartheta_i^{(-d-1)/2}\right).$$

Cooper and Spencer show in [CS06] that $f(\vartheta) = p(\mathbf{z}+\mathbf{e}',\vartheta)-p(\mathbf{z},\vartheta) = \mathcal{O}(\|\mathbf{z}\|^{-d-1})$ and, by using a monotonicity argument similar to unimodal sequences, also

$$\sum_{i=0}^{s}(-1)^i f(\vartheta_i) = \mathcal{O}\left(\max_i |f(\vartheta_i)|\right) = \mathcal{O}\left(\|\mathbf{z}\|^{-d-1}\right).$$

Therefore,

$$
\begin{aligned}
\left|\sum_{\substack{i=0\\t_i<T}}^{s}(-1)^i\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)\right| &\leq \left|\sum_{\substack{i=0\\t_i<T}}^{s}(-1)^i f(\vartheta_i)\right| + \sum_{\substack{i=0\\t_i<T}}^{s}\left|\nabla_{\mathbf{e}'}(\mathbf{z},\vartheta_i)-p(\mathbf{z}+\mathbf{e}',\vartheta_i)+p(\mathbf{z},\vartheta_i)\right| \\
&\leq \mathcal{O}\left(\|\mathbf{z}\|^{-d-1}\right)+\sum_{\vartheta=t-T}^{\infty}\|\mathbf{z}\|^{-2}\mathcal{O}\left(\vartheta^{(-d-1)/2}\right) \\
&= \mathcal{O}\left(\|\mathbf{z}\|^{-d-1}\right)+\mathcal{O}\left(\|\mathbf{z}\|^{-2}(t-T)^{(-d+1)/2}\right) \\
&= \mathcal{O}\left(\|\mathbf{z}\|^{-d-1}\log^{d-1}\|\mathbf{z}\|\right)
\end{aligned}
$$

by our choice of T. Together with (2.12), we thus have

$$
\begin{aligned}
\left|\sum_{i=0}^{s}(-1)^i\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)\right| &\leq \left|\sum_{\substack{i=0\\t_i<T}}^{s}(-1)^i\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)\right| + \sum_{\substack{i=0\\t_i\geq T}}^{s}\left|\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)\right| \\
&= \mathcal{O}\left(\|\mathbf{z}\|^{-d-1}\log^{d-1}\|\mathbf{z}\|\right).
\end{aligned}
$$

For fixed $\mathbf{z}$ and $d \geq 2$, we can bound similarly

$$
\begin{aligned}
\left|\sum_{i=0}^{s}(-1)^i\nabla_{\mathbf{e}'}(\mathbf{z},t-t_i)\right| &\leq \left|\sum_{i=0}^{s}(-1)^i f(\vartheta_i)\right| + \sum_{i=0}^{s}\left|\nabla_{\mathbf{e}'}(\mathbf{z},\vartheta_i)-p(\mathbf{z}+\mathbf{e}',\vartheta_i)+p(\mathbf{z},\vartheta_i)\right| \\
&\leq \max_i \underbrace{|f(\vartheta_i)|}_{\leq 2} + \sum_{\vartheta=t-T}^{\infty}\mathcal{O}\left(\vartheta^{(-d-1)/2}\right) = \mathcal{O}(1),
\end{aligned}
$$

from which follows (2.11) and the claim. □

This is much more general than our one-dimensional result from last section, however it is quite a tricky job to actually determine the constant c_d for each d. The value for $d = 2$ is known due to the work of Doerr and Friedrich and they also showed that the rotor sequences, i. e. the functions $n_{\mathbf{x}}$, can make a significant difference for $d > 1$.

Proposition 2.4.3 (c.f. [DF09])
For the constant c_2 of Theorem 2.4.2 we have,

- *if all vertices may have different rotor sequences,* $7.873 \leq c_2 \leq 8.026$,
- *if all vertices have the same circular rotor sequence,* $7.832 \leq c_2 \leq 7.985$,
- *and if all vertices have the same non-circular rotor sequence,* $7.286 \leq c_2 \leq 7.439$.

3. Internal diffusion limited aggregation using quasirandom walks

3.1. Basic properties

Definition 3.1.1 (Internal diffusion limited aggregation)
For $d \in \mathbb{N}$ and $n \geq 1$, let $A_0 = \emptyset$ and $A_n \subseteq \mathbb{Z}^d$ be the set we obtain from taking A_{n-1}, starting a simple random walk in 0, stopping it when it first leaves A_{n-1}, and adding the endpoint to A_{n-1}. Formally, we have $A_n = A_{n-1} \cup \left\{X_{\tau_{n-1}}\right\}$, where $X = (X_0, X_1, \dots)$ is a random walk on $\mathbb{Z}^d$ and $\tau_n = \tau_{A_n^c}$ is the hitting time of A_n^c.

The $\mathcal{P}(\mathbb{Z}^d)$-valued Markov chain $A_0, A_1, \dots$ is called internal diffusion limited aggregation (IDLA) *on $\mathbb{Z}^d$. The elements of A_n are said to be* occupied.

The IDLA model, first introduced by Diaconis and Fulton in [DF91], has probably been one of the most discussed aggregation models in recent time. We put chips onto the origin and let them walk randomly until they find an empty space they can occupy; one could hardly think of any simpler algorithm. Thus, it is even more interesting that this model exhibits many properties which can be a useful tool in modeling a symmetric, centered growth. In [LBG92] it is shown that with probability one we have $n^{1/d} A_n \to B_1$ as $n \to \infty$, where B_1 is the Euclidean ball with unit radius in $\mathbb{R}^d$. The rate of convergence is further bounded in [Law95]. Note that IDLA is well-defined by Proposition 2.1.6, which guarantees $\tau_X(A_n^c) < \infty$ almost surely.

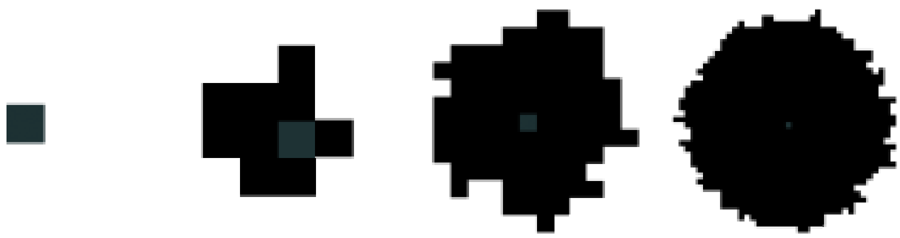


Figure 3.1.: Possible values for the sets A_1, A_{10}, A_{100}, and $A_{1,000}$ (in different scale).

Example 3.1.2 (IDLA on $\mathbb{Z}^2$)
Most frequently, we are going to look at IDLA in two dimensions. This is still simple enough to be displayed easily but already a bit more complex than IDLA on $\mathbb{Z}^1$, where there are

always only two possible directions of growing. Although we want to cover results for all dimensions, the most interesting examples (as shown in Appendix A) will be two-dimensional.

For $n = 1$, 10, 100, and 1000, Figure 3.1 shows possible sets A_n of occupied fields after running n chips through the IDLA model on $\mathbb{Z}^2$. A circular structure for higher n is easily noticeable.

One can check that a model, where we place n (indiscernible) chips at the origin and let each of them perform independent random walks, stopping the chips as soon as they hit an empty field, will eventually yield the same probabilities for a specified A_n as in the definition above. In addition, IDLA is easily expandable by placing different numbers of chips on different fields.

After discussing quasirandom walks in such detail in the last chapter, the next step seems obvious: We want to use the rotor machine for aggregation. Therefore, we again install rotors on each field of $\mathbb{Z}^d$, which are rotated each time a chip leaves the corresponding field. Then, as in the random IDLA, we place a chip at the origin, let it run until it finds an empty spot, and repeat the whole procedure.

Definition 3.1.3 (Internal diffusion limited aggregation for quasirandom walks)
On the graph $\mathbb{Z}^d$, let $\sigma_{0,0}$ be some initial rotor state and $A_0 = \emptyset$. For $n \geq 1$, let $X = (X_0, X_1, \ldots, X_{\tau_{n-1}})$ be a quasirandom walk with $X_0 = 0$, initial rotor state $\sigma_{n,0}$, and the hitting time $\tau_n = \tau_{A_n^c}$. Then, set $A_n = A_{n-1} \cup \left\{X_{\tau_n}\right\}$ and let $\sigma_{n+1,0} = \sigma_{n,\tau_n}$ be the rotor state after the last move. We call A_0, A_1, ... quasirandom internal diffusion limited aggregation *on $\mathbb{Z}^d$.*

We should be aware that IDLA for quasirandom walks is not obviously well-defined, as there could possibly be initial rotor configurations leading to $\tau_n = \infty$ for some n. However, from Proposition 2.1.7 we know that this cannot be the case, as A_n is a finite subset of $\mathbb{Z}^d$ for all n.

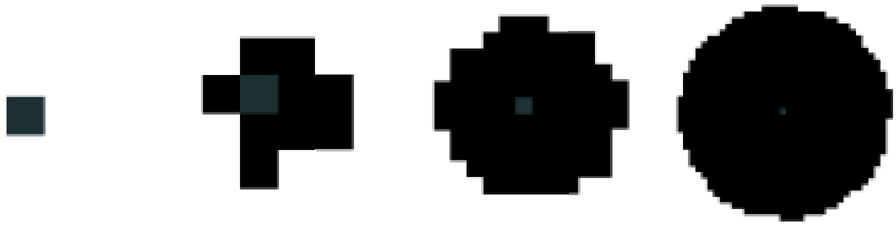


Figure 3.2.: Possible values for the sets A_1, A_{10}, A_{100}, and $A_{1,000}$ (in different scale).

Example 3.1.4 (Quasirandom IDLA on $\mathbb{Z}^2$)
In Figure 3.2 we see the sets A_n of n occupied fields for the IDLA model on $\mathbb{Z}^2$ with $n = 1$, 10, 100, and 1,000. The initial rotor state was $\rightarrow$ for all fields and the rotor sequence used was $(\rightarrow, \downarrow, \leftarrow, \uparrow)$. Also, compare the circularity of this example to Figure 3.1.

We also give a comparison of quasirandom and random IDLA in Figure 3.3 after 100,000 chips, where the initial configuration for the rotor model is the same as before. The gray area marks are all fields which lie on the path of the 100,000th chip on its way to the circle boundary. This illustrates the high level of "randomness" we experience from quasirandom walks.

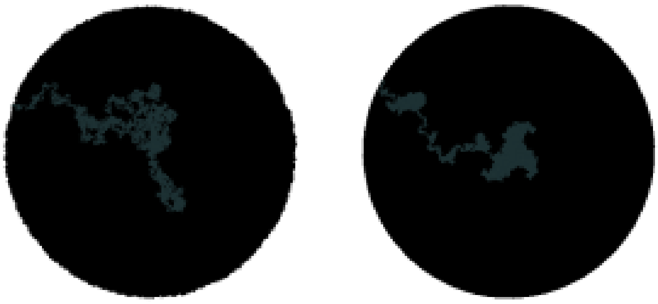


Figure 3.3.: Possible values for the sets $A_{100,000}$ using random (left) and quasirandom (right) IDLA.

As done in this example, we will from now on sometimes use arrow-notation to indicate rotor directions instead of $n_{\mathbf{x}}(\sigma_t(\mathbf{x}))$. Results on the general behavior of quasirandom IDLA certainly do not depend on the exact representation of rotors via σ_t and the bijection $n_{\mathbf{x}}$ and so we will use these notations interchangeably. For any vertex, we therefore define an initial direction and a rotor sequence, according to which we update the rotor. In almost all cases, the initial direction will be *global*, i. e. equal for all vertices.

Note that in $\mathbb{Z}^2$, there are effectively only three principal rotor sequences for a fixed global initial direction: $(\rightarrow, \downarrow, \leftarrow, \uparrow)$, $(\rightarrow, \downarrow, \uparrow, \leftarrow)$, and $(\rightarrow, \leftarrow, \downarrow, \uparrow)$. All other rotor sequences produce equal results up to isometry.

Also, for quasirandom rotor-router aggregation, we can place n chips on the origin and let each of them perform quasirandom walks. Then, iteratively, we take an arbitrary position where there is more than one chip and let the top chip perform one quasirandom step. We stop after there is no more than one chip on each field.

It is easy to show that this process will create the same sets A_n as defined above, independent of the choice of positions from where the next chip is moved. Even a mix of adding chips and routing them would be equivalent to quasirandom IDLA. This is said to be the *Abelian* property of quasirandom walks. In addition, we could allow that not necessarily all chips have to start from the origin, resulting in arbitrary initial distributions of chips. However, we will continue to focus on quasirandom IDLA starting from 0.

Levine and Peres slightly generalize the concept of quasirandom walks in [LP08] by placing arbitrary infinite stacks of rotor-directions $\sigma^{\mathbf{x}} = (\sigma_0^{\mathbf{x}}, \sigma_1^{\mathbf{x}}, \ldots) \in (\mathbb{Z}_{2d})^{\mathbb{N}}$ on each point $\mathbf{x}$, where the i-th chip leaving $\mathbf{x}$ will travel into direction $\sigma_i^{\mathbf{x}}$ instead of some cyclically updated one. However, we will require a certain structure from those stacks, namely that for any possible direction $\delta \in \mathbb{Z}_{2d}$, $m > 0$ and $\mathbf{x} \in \mathbb{Z}^d$

$$\left| \#\{i \leq m : \sigma_i^{\mathbf{x}} = \delta\} - \frac{m}{2d} \right| \leq D$$

for a constant $D > 0$. Subsequently, we say D is the *stack discrepancy* of the rotor-router model. Note that this condition holds for the (original) cyclical quasirandom walks with $D = 1$.

The main goal of this chapter will be to prove circularity for the resulting "blobs" of quasirandom IDL. However, we first have to specify the term *circularity* when working with subsets of $\mathbb{Z}^d$, which is not a uniquely solvable task. We have already seen scaled convergence to a unit ball as one specific measure of circularity, though the following certainly is much stronger.

Definition 3.1.5 (Outer and inner radius)
Let $\{0\} = A_1 \subseteq A_2 \subseteq \cdots \subseteq \mathbb{Z}^d$ *be a quasirandom or random IDLA. Then we define* ($\|\cdot\|$ *being the 2-norm) the* outer radius *of* A_n *as*

$$R_n = \inf\{R \geq 0 : A_n \subseteq B_R\}$$

and the inner radius

$$r_n = \sup\{r \geq 0 : B_r \cap \mathbb{Z}^d \subseteq A_n\},$$

where B_r *is the* d*-dimensional ball around* 0 *of radius* r*. We call* $|R_n - r_n|$ *the* radius difference.

A high circularity therefore can be characterized by small $|R_n - r_n|$ as $n \to \infty$. Note that, using this definition, $R_n > r_n$ does not necessarily hold as we restrict our conditions on $A_n \subseteq \mathbb{Z}^n$. Still, using the whole filled area would change the radius difference only by a constant.

Figure 3.4 displays the radius differences for each $n \leq 100,000$ for the two models that are also shown in the example of Figure 3.3. The radius difference is clearly much larger and faster growing for random IDLA. The maximum difference in this example for random aggregation is 7.10 and 1.72 for the quasirandom version.

For quasirandom IDLA, all empirical results suggest that the radius differences for large n are remarkably lower. As conjectured in [Kle05], we expect $|R_n - r_n| < \varrho_d$ for some constant ϱ_d only depending on the dimension of the model. In the next section, we are going to further explore the circularity of quasirandom aggregation.

3.2. Convergence to a ball

For $A \subseteq \mathbb{Z}^d$, define $A^{\square} = A + [-1/2, 1/2]^d \subseteq \mathbb{R}^d$ as Minkowski-sum and let ω_d be the volume of the d-dimensional unit ball B_1. The first result on the circularity of quasirandom IDLA

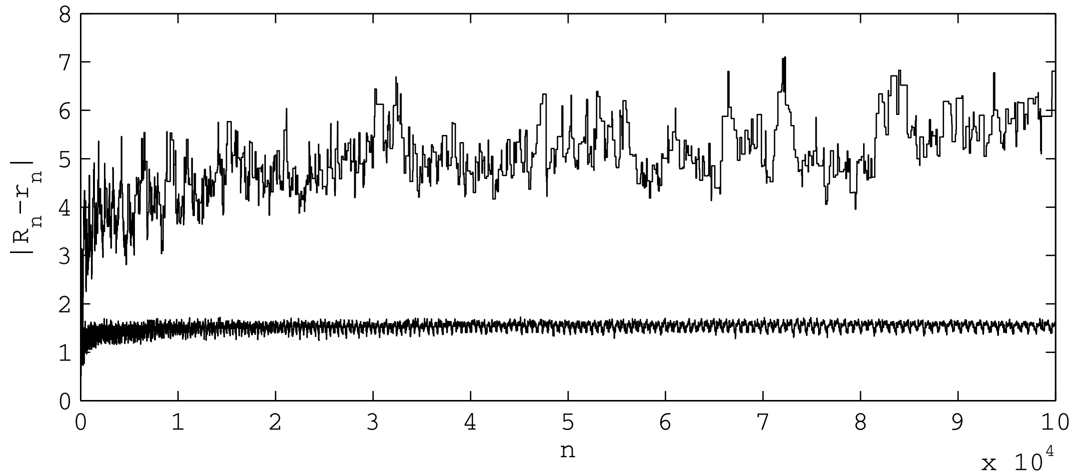


Figure 3.4.: Radius differences $|R_n - r_n|$, $1 \leq n \leq 100000$, for random (top) and quasirandom (bottom) IDLA on $\mathbb{Z}^2$.

was found by Levine and Peres, showing that the volume of the symmetric difference of $A_n^\square$ scaled to unit diameter and B_1 converges to zero.

Theorem 3.2.1 (Convergence of quasirandom IDLA in $\mathscr{L}$, [LP08])
Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq \mathbb{Z}^d$ be the sets of occupied vertices produced by quasirandom IDLA on $\mathbb{Z}^d$ for some $d \in \mathbb{N}$ and arbitrary rotor stacks $\sigma^{\mathbf{x}}$ with finite stack discrepancy $\leq D$. Then

$$\mathscr{L}\left((\omega_d/n)^{1/d} A_n^\square \triangle B_1\right) \to 0$$

as $n \to \infty$, where $\mathscr{L}$ denotes the d-dimensional Lebesgue measure and $A \triangle B$ denotes the symmetric difference of sets A and B.

However, the authors were able to improve this result substantially. The next theorem on quasirandom IDLA gives the best bounds known so far, though we will first show two lemmas we are going to need during the proof.

Lemma 3.2.2 (c.f. [LP09])
Let H_1, H_2 be linear half-spaces in $\mathbb{Z}^d$ and τ_i be the hitting time of H_i for a random walk X. For $\mathbf{x} \notin H_1 \cup H_2$, we have

$$\mathbb{P}_{\mathbf{x}}(\tau_1 > \tau_2) \leq \frac{7}{2} \frac{h_1 + 1}{h_2} \left(1 + \frac{1}{2h_2}\right)^2$$

where $h_i = \operatorname{dist}(\mathbf{x}, H_i)$.

Proof. The result is trivial if one of the half-spaces contains the other. Otherwise, let $\tilde{H}_1$ be the half-space H_1^c shifted by $2h_2$ in the direction of $\mathbf{x}$ and let $\tilde{\tau}_1$ be the hitting time of $H_1 \cup \tilde{H}_1$. Define $\tilde{H}_2$ and $\tilde{\tau}_2$ analogously. Write $M_t = \operatorname{dist}^{\pm}(X_t, \partial H_1)$ as the signed distance with $\text{dist}^{\pm}(\mathbf{x}, \partial H_1) = h_1$, then M_t is a martingale with bounded increments. As $\mathbb{E}_{\mathbf{x}}[\tilde{\tau}_1] < \infty$ we can use the optional stopping theorem [Kle08, p. 210ff.] to obtain

$$h_1 = \mathbb{E}_{\mathbf{x}}[M_0] = \mathbb{E}_{\mathbf{x}}[M_{\tilde{\tau}_1}] \geq 2h_2 \mathbb{P}_{\mathbf{x}}(X_{\tilde{\tau}_1} \in \tilde{H}_1) + (-1)\mathbb{P}_{\mathbf{x}}(X_{\tilde{\tau}_1} \in H_1)$$

and therefore

$$\mathbb{P}_{\mathbf{x}}(X_{\tilde{\tau}_1} \in \tilde{H}_1) \leq \frac{h_1 + 1}{2h_2 + 1} \leq \frac{h_1 + 1}{2h_2}. \tag{3.1}$$

Also, $dM_t^2 - t$ is a martingale with bounded increments, thus by optional stopping

$$\begin{aligned} 0 \leq \mathbb{E}_{\mathbf{x}}[dM_0^2] = \mathbb{E}_{\mathbf{x}}[dM_{\tilde{\tau}_1}^2 - \tilde{\tau}_1] = d\mathbb{E}_{\mathbf{x}}[M_{\tilde{\tau}_1}^2] - \mathbb{E}_{\mathbf{x}}[\tilde{\tau}_1] \Leftrightarrow \\ \mathbb{E}_{\mathbf{x}}[\tilde{\tau}_1] \leq d(2h_2 + 1)^2 \mathbb{P}_{\mathbf{x}}(X_{\tilde{\tau}_1} \in \tilde{H}_1) + d\mathbb{P}_{\mathbf{x}}(X_{\tilde{\tau}_1} \in H_1) \\ \leq d(h_1 + 1)(2h_2 + 1)\left(1 + \frac{1}{2h_2}\right) + d. \end{aligned} \tag{3.2}$$

Let $\tau = \tilde{\tau}_1 \wedge \tilde{\tau}_2 = \min(\tilde{\tau}_1, \tilde{\tau}_2)$. Define $p = \mathbb{P}_{\mathbf{x}}(\tau = \tilde{\tau}_2)$ and $D_t = \text{dist}^{\pm}(X_t, \partial H_2)$. One can show that $N_t = \frac{d}{2}\left(D_t^2 + (2h_2 - D_t)^2\right) - t$ is a martingale and thus

$$\begin{aligned} dh_2^2 = \mathbb{E}_{\mathbf{x}}[N_0] = \mathbb{E}_{\mathbf{x}}[N_\tau] \geq p\frac{d}{2}(2h_2)^2 + (1-p)dh_2^2 - \mathbb{E}_{\mathbf{x}}[\tau] \\ \geq (1+p)dh_2^2 - \mathbb{E}_{\mathbf{x}}[\tau]. \end{aligned}$$

By (3.2) we have

$$p \leq \frac{\mathbb{E}_{\mathbf{x}}[\tau]}{dh_2^2} \leq 2\frac{h_1 + 1}{h_2}\left(1 + \frac{1}{2h_2}\right)^2 + \frac{1}{h_2^2} \leq 3\frac{h_1 + 1}{h_2}\left(1 + \frac{1}{2h_2}\right)^2$$

and by (3.1) Lemma 3.2.2 follows from

$$\mathbb{P}_{\mathbf{x}}(\tau_1 > \tau_2) \leq p + \mathbb{P}_{\mathbf{x}}(X_{\tilde{\tau}_1} \in \tilde{H}_1) \leq \frac{7}{2}\frac{h_1 + 1}{h_2}\left(1 + \frac{1}{2h_2}\right)^2.$$

□

Lemma 3.2.3 (by Holroyd and Propp, published in [LP09])
Let $G = (V, E)$ be a finite connected graph, let $Y \subseteq Z \subseteq V$ arbitrary, and let $s : V \to \mathbb{N}_{\geq 0}$. Let $H_W(s, Y)$ be the expected number of particles having stopped in Y if $s(x)$ particles start from each vertex x and perform independent simple random walks stopped on first hitting Z. Let $H_R(s, Y)$ be the number of particles having stopped in Y if $s(x)$ particles start from each vertex x and perform rotor-router quasirandom walks stopped on first hitting Z. Define $H(x) = H_W(\mathbf{1}_{\{x\}}, Y)$, then

$$|H_R(s, Y) - H_W(s, Y)| \leq \sum_{u \notin Z} \sum_{v \in N_G(u)} |H(u) - H(v)|$$

independent of s and the initial rotor configuration σ_0 and n.

Proof. For each vertex $u \notin \mathbb{Z}$, assign the *weight* $w(u, n_u(0)) = 0$ for a rotor pointing from u to $n_u(0)$ and $w(u, n_u(i)) = H(u) - H(n(i)) + w(u, n_u(i-1))$ for a rotor pointing from u to $n_u(i)$ and $i \in \mathbb{Z}_{d(u)} - \{0\}$, where n_u is the bijection between $\mathbb{Z}_{d(u)}$ and $N(u)$. Further, assign weight $H(u)$ to the particle located at u. The sum of rotor and particle weights is now an invariant in any configuration under the operation of moving a chip and updating the rotor. Initially, the sum of all particle weights is $H_W(s, Y)$ and after all quasirandom walks have stopped it is $H_R(s, Y)$. The difference can only be the change in rotor weights, which is bounded by the right sum as given. □

Theorem 3.2.4 (Spherical bounds for quasirandom IDLA, [LP09])
Let $(A_n)_{n\geq 1}$ be as in Theorem 3.2.1 (using arbitrary rotor stacks with finite stack discrepancy D). Then there exist constants c and c' only depending on d, such that for all $n \in \mathbb{N}$

$$B_{r-c\log r} \subseteq A_n^{\square} \subseteq B_{r\left(1+c' r^{-1/d}\log r\right)} \tag{3.3}$$

where $r = (n/\omega_d)^{1/d}$.

Proof. The proof is quite lengthy, so we will show the basic ideas at that point and refer to [LP09] for details. We will also only consider the case $d \geq 2$ and give a stronger result for $d = 1$ later on. For a simple random walk X on $\mathbb{Z}^d$ and for $\mathbf{x} \in \mathbb{Z}^d$, we define the function $g_n(\mathbf{x}) = \mathbb{E}_0[\#\{t \leq n : X_t = \mathbf{x}\}]$ and subsequently the *discrete harmonic Green's function*

$$g(\mathbf{x}) = \begin{cases} \mathbb{E}_0[\#\{t \geq 0 : X_t = \mathbf{x}\}] & \text{for } d \geq 3 \\ \lim_{n\to\infty}(g_n(\mathbf{x}) - g_n(0)) & \text{for } d = 2. \end{cases} \tag{3.4}$$

We need to differentiate between those two cases as recurrence of random walks in two dimensions would have implied infinite values for $g(\mathbf{x})$. For an edge $(\mathbf{x}, \mathbf{y})$ and a function f, we further write

$$\begin{aligned} \nabla f(\mathbf{x}, \mathbf{y}) &= f(\mathbf{y}) - f(\mathbf{x}) \\ \operatorname{div} f(\mathbf{x}) &= \frac{1}{2d}\sum_{\mathbf{y}\in N(\mathbf{x})} f(\mathbf{x}, \mathbf{y}) \\ \Delta f(\mathbf{x}) = \operatorname{div}\nabla f(\mathbf{x}) &= \frac{1}{2d}\sum_{\mathbf{y}\in N(\mathbf{x})} f(\mathbf{y}) - f(\mathbf{x}). \end{aligned}$$

The operator Δ is also called the *discrete Laplacian*. One can easily verify that $\Delta g = -\mathbf{1}_{\{0\}}$. Finally, for fixed n, define the *odometer function* of the rotor model with n chips by $u(\mathbf{x}) =$ total number of chips having moved from $\mathbf{x}$.

The first part of the proof will show the left inclusion of (3.3). Our main goal is to guarantee $u(\mathbf{x}) > 0$ for certain $\mathbf{x}$, as this implies that $\mathbf{x}$ is occupied. Set $\tilde{\gamma}_d(\mathbf{x}) = \|\mathbf{x}\|^2 + ng(\mathbf{x})$ and

$$\gamma_d(\mathbf{x}) = \tilde{\gamma}_d(\mathbf{x}) - \tilde{\gamma}_d(\lfloor r \rfloor \mathbf{e}_1), \tag{3.5}$$

where $\mathbf{e}_1$ is the first standard basis vector in $\mathbb{Z}^d$. By estimating g as in [Law96, pp. 31 and 38] and doing some additional calculus, one can show the following bounds for γ_d.

$$\gamma_d(\mathbf{x}) \geq (r - \|\mathbf{x}\|)^2 + \mathcal{O}\left(\frac{r^d}{\|\mathbf{x}\|^d}\right) \qquad \text{for all } \mathbf{x} \in \mathbb{Z}^d, \tag{3.6}$$

$$\gamma_d(\mathbf{x}) = \mathcal{O}(1) \qquad \text{uniformly in } r \text{ for } \mathbf{x} \in B_{r+1} - B_{r-1}, \text{ and} \tag{3.7}$$

$$\gamma_d(\mathbf{x}) > \frac{r^2}{4} \qquad \text{for all } \mathbf{x} \in B_{r/3} \text{ and } r \text{ sufficiently large.} \tag{3.8}$$

Next, for an edge $(\mathbf{x}, \mathbf{y})$, let κ be the net number of crossings from $\mathbf{x}$ to $\mathbf{y}$. Then we can easily see that

$$\nabla u(\mathbf{x}, \mathbf{y}) = -2d\kappa(\mathbf{x}, \mathbf{y}) + R(\mathbf{x}, \mathbf{y}) \tag{3.9}$$

for some R with $R(\mathbf{x}, \mathbf{y}) \leq C_0$ and C_0 a constant depending on the maximum stack discrepancy. Also there exists a C_1, such that for any finite set $0 \notin S \subset \mathbb{Z}^d$

$$\sum_{\mathbf{y} \in S} \|\mathbf{y}\|^{1-d} \leq C_1 \operatorname{diam}(S). \tag{3.10}$$

Let $Y_t = X_{t \wedge \tau_{B_r^c}}$ be a random walk stopped on exiting B_r, where $t_1 \wedge t_2 = \min(t_1, t_2)$ for all $t_1, t_2 \in \mathbb{R}$. Define $G = G_{B_r}$ to be its *Green's function* by $G(\mathbf{x}, \mathbf{y}) = \mathbb{E}_{\mathbf{x}}[\#\{t \geq 0 : Y_t = \mathbf{y}\}]$. Subsequently, we can use (3.10) to prove that for some constant C_2 and for any $\varrho \geq 1$ and $\mathbf{x} \in B_r$

$$\sum_{\substack{\mathbf{y} \in B_r \\ \|\mathbf{x}-\mathbf{y}\| \leq \varrho}} \sum_{\mathbf{z} \in N(\mathbf{y})} |G(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{z})| \leq C_2 \varrho. \tag{3.11}$$

Now, let $\mathbf{x} \in B_r$, $\varrho = r + 1 - \|\mathbf{x}\|$, and $\mathcal{S}_k^* = \left\{\mathbf{y} \in B_r : 2^k < \|\mathbf{x} - \mathbf{y}\| \leq 2^{k+1}\varrho\right\}$. We can show, as a corollary to Lemma 3.2.2, that there exists a C_3 such that

$$\mathbb{P}_{\mathbf{x}}(\vartheta_k < T) \leq C_3 2^{-k}, \tag{3.12}$$

where ϑ_k is the hitting time of $\mathcal{S}_k^*$ and T the exit time from B_r. Looking back at κ, we note that there can be only one chip on $\mathbf{x}$. Therefore, $2d \operatorname{div} \kappa(\mathbf{x}) \geq -1$ for $\mathbf{x} \neq 0$ and $2d \operatorname{div} \kappa(0) = n - 1$, which together with (3.9) yields

$$\begin{aligned} \Delta u(\mathbf{x}) &\leq 1 + \operatorname{div} R(\mathbf{x}) \\ \Delta u(0) &= 1 - n + \operatorname{div} R(0). \end{aligned} \tag{3.13}$$

We define

$$f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[u(X_T)] - \mathbb{E}_{\mathbf{x}}[T] + n\mathbb{E}_{\mathbf{x}}[\#\{k < T : X_k = 0\}].$$

One can check that $\Delta f(\mathbf{x}) = 1$ for $\mathbf{x} \in B_r - \{0\}$ as well as $\Delta f(0) = 1 - n$. Because $f \geq 0$ on $\partial B_r = \{\mathbf{z} \in B_r^c : \exists \mathbf{y} \in B_r : \mathbf{z} \in N(\mathbf{y})\}$ we get $f(\mathbf{x}) = \Omega(1)$ on B_r. With (3.7) it follows that

$f \geq \gamma - C_4$ for some constant C_4. Splitting up, we have

$$u(\mathbf{x}) - \mathbb{E}_{\mathbf{x}}[u(X_T)] = \sum_{k\geq 0} \mathbb{E}_{\mathbf{x}}\left[u(X_{k\wedge T}) - u(X_{(k+1)\wedge T})\right]$$

and also

$$\mathbb{E}_{\mathbf{x}}\left[u(X_{k\wedge T}) - u(X_{(k+1)\wedge T})|\mathscr{F}_{k\wedge T}\right] = \mathbf{1}_{\{T>k\}}\left(u(X_k) - \frac{1}{2d}\sum_{\mathbf{y}\in N(X_k)} u(\mathbf{y})\right) = -\Delta u(X_k)\mathbf{1}_{\{T>k\}}.$$

By (3.13) we obtain

$$\begin{aligned} u(\mathbf{x}) - \mathbb{E}_{\mathbf{x}}[u(X_T)] &\geq \sum_{k\geq 0} \mathbb{E}_{\mathbf{x}}\left[\mathbf{1}_{\{T>k\}}\left(n\mathbf{1}_{\{X_k=0\}} - 1 - \operatorname{div} R(X_k)\right)\right] \\ &= n\mathbb{E}_{\mathbf{x}}[\#\{k<T : X_k = 0\}] - \mathbb{E}_{\mathbf{x}}[T] - \sum_{k\geq 0} \mathbb{E}_{\mathbf{x}}[\mathbf{1}_{\{T>k\}} \operatorname{div} R(X_k)] \end{aligned}$$

and thus, using that $X_k \in B_r$ for $k < T$ and $R \leq C_0$,

$$\begin{aligned} u(\mathbf{x}) - f(\mathbf{x}) &\geq -\frac{1}{2d}\sum_{k\geq 0}\mathbb{E}_{\mathbf{x}}\left[\sum_{\mathbf{z}\in N(X_k)} \mathbf{1}_{\{T>k\}} R(X_k, \mathbf{z})\right] \\ &\geq -\frac{1}{2d}\sum_{k\geq 0}\mathbb{E}_{\mathbf{x}}\left[\sum_{\substack{\mathbf{y},\mathbf{z}\in B_r\\ \mathbf{z}\in N(\mathbf{y})}} \mathbf{1}_{\{X_k=\mathbf{y}\}\cap\{T>k\}} R(\mathbf{y},\mathbf{z}) + \sum_{\substack{\mathbf{y}\in B_r, \mathbf{z}\notin B_r\\ \mathbf{z}\in N(\mathbf{y})}} \mathbf{1}_{\{X_k=\mathbf{y}\}\cap\{T>k\}} R(\mathbf{y},\mathbf{z})\right] \\ &\geq -\frac{1}{2d}\sum_{k\geq 0}\sum_{\substack{\mathbf{y},\mathbf{z}\in B_r\\ \mathbf{z}\in N(\mathbf{y})}} \mathbb{P}_{\mathbf{x}}(X_{k\wedge T} = \mathbf{y})R(\mathbf{y},\mathbf{z}) - 2dC_0 \end{aligned}$$

where the last step follows from the fact that the expected time spent at the boundary of B_r before T is less than $2d$. By summing over k and using

$$\mathbb{P}_{\mathbf{x}}(X_{k\wedge T} = \mathbf{y})R(\mathbf{y},\mathbf{z}) = \frac{\mathbb{P}_{\mathbf{x}}(X_{k\wedge T} = \mathbf{y}) - \mathbb{P}_{\mathbf{x}}(X_{k\wedge T} = \mathbf{z})}{2}R(\mathbf{y},\mathbf{z}),$$

one can further show

$$u(\mathbf{x}) \geq f(\mathbf{x}) - \frac{C_0}{4d}\sum_{\mathbf{y}\in B_r} \mathfrak{G}(\mathbf{x},\mathbf{y}) - 2dC_0, \tag{3.14}$$

writing $\mathfrak{G}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{z}\in N(\mathbf{y})\cap B_r} |G(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{z})|$. We will now look for upper bounds for the sum

$$\begin{aligned}\sum_{\mathbf{y}\in B_r} \mathfrak{G}(\mathbf{x},\mathbf{y}) &= \sum_{\|\mathbf{y}-\mathbf{x}\|\le\varrho} \mathfrak{G}(\mathbf{x},\mathbf{y}) + \sum_{k=0}^{\lceil \log_2(r/\varrho)\rceil} \sum_{\mathbf{y}\in\mathcal{S}_k^*} \mathfrak{G}(\mathbf{x},\mathbf{y}) \\ &= \sum_{\|\mathbf{y}-\mathbf{x}\|\le\varrho} \mathfrak{G}(\mathbf{x},\mathbf{y}) + \sum_{k=0}^{\lceil \log_2(r/\varrho)\rceil} \sum_{\mathbf{y}\in\mathcal{S}_k^*} \sum_{\mathbf{w}\in\partial\mathcal{S}_k^*} \mathbb{P}_{\mathbf{x}}(X_{\hat{\tau}_k\wedge T} = \mathbf{w})\mathfrak{G}(\mathbf{w},\mathbf{y}),\end{aligned}$$

where $\hat{\tau}_k$ denotes the hitting time of $\partial\mathcal{S}_k^*$. As $\|\mathbf{y}-\mathbf{w}\| \le 3\cdot 2^k\varrho$, by (3.11) and (3.12) we get

$$\begin{aligned}\sum_{\mathbf{y}\in B_r} \mathfrak{G}(\mathbf{x},\mathbf{y}) &\le C_2\varrho + \sum_{k=0}^{\lceil \log_2(r/\varrho)\rceil} 3C_2 2^k\varrho \sum_{\mathbf{w}\in\partial\mathcal{S}_k^*} \mathbb{P}_{\mathbf{x}}(X_{\hat{\tau}_k\wedge T} = \mathbf{w}) \\ &\le C_2\varrho + \sum_{k=0}^{\lceil \log_2(r/\varrho)\rceil} (3C_2 2^k\varrho)(C_3 2^{-k}) \le C_5\varrho\log\frac{r}{\varrho}.\end{aligned}$$

Inserting this into (3.14), together with (3.6), finally yields

$$u(\mathbf{x}) \ge (r-\|\mathbf{x}\|)^2 - \frac{C_0C_5}{4d}(r+1-\|\mathbf{x}\|)\log\frac{r}{r+1-\|\mathbf{x}\|} + \mathcal{O}\left(\frac{r^d}{\|\mathbf{x}\|^d}\right),$$

which is strictly positive for $r/3 \le \|\mathbf{x}\| < r - c\log r$, and for $\|\mathbf{x}\| < r/3$ we have $u(\mathbf{x}) > r^2/4 - \mathcal{O}(r)$ by (3.8). This shows the left inclusion of (3.3), as there has to be a chip on all fields where $u > 0$.

Fix constants $\varrho \ge r$, $h \ge 1$, and define $\mathcal{S}_k = \{\mathbf{x}\in\mathbb{Z}^d : k \le \|\mathbf{x}\| < k+1\}$. Thus, $B_\varrho = \mathcal{S}_0 \cup\cdots\cup \mathcal{S}_{\varrho-1}$. We then use the Abelian property of quasirandom aggregation and place all n chips on the origin, from where they perform a quasirandom walk, being stopped if they reach an empty space *or* the shell $\mathcal{S}_{\varrho+h}$. Let $N_{\varrho+h}$ be the number of particles that reach $\mathcal{S}_{\varrho+h}$ and let N_ϱ be the number of particles that at least once reach $\mathcal{S}_\varrho$ (obviously $N_{\varrho+h} \le N_\varrho$).

In the setting of Lemma 3.2.3 set $G = B_{\varrho+h+1}$ and $Z = \mathcal{S}_{\varrho+h}$. Fix a $\mathbf{y}\in\mathcal{S}_{\varrho+h}$ and set $Y = \{\mathbf{y}\}$. For $\mathbf{x}\in\mathcal{S}_\varrho$, let $s(\mathbf{x})$ be the number of particles at $\mathbf{x}$ if each particle following a quasirandom walk would be stopped after reaching an unoccupied field or the shell $\mathcal{S}_\varrho$. We also define $H(\mathbf{x}) = \mathbb{P}_{\mathbf{x}}(X_T = y)$, where T is the hitting time of $\mathcal{S}_{\varrho+h}$ for a simple random walk. It is not difficult to show that there exists a constant J such that

$$H_W(s,\mathbf{y}) = \sum_{x\in\mathcal{S}_\varrho} s(\mathbf{x})H(\mathbf{x}) \le \frac{JN_\varrho}{h^{d-1}}.$$

We can apply Lemma 3.2.3 and get (for some constant J')

$$H_R(s,\mathbf{y}) \leq \frac{JN_\varrho}{h^{d-1}} + \sum_{\mathbf{w}\in B_{\varrho+h}} \sum_{\mathbf{v}\in N(\mathbf{w})} |H(\mathbf{w}) - H(\mathbf{v})| \leq \frac{JN_\varrho}{h^{d-1}} + J' \log h$$

where the last step is shown in more detail in [LP09].

Inductively, define $\varrho(i+1) = \min\left\{\varrho(i) + N_{\varrho(i)}^{2/(2d-1)}, \min\{\varrho > \varrho(i) : N_\varrho \leq N_{\varrho(i)}/2\}\right\}$ for $i \geq 1$ where $\varrho(0) = r$. For $0 \leq h < \varrho(i+1) - \varrho(i)$ we have

$$h^{d-1}\log h \leq N_{\varrho(i)}^{(2d-2)/(2d-1)} \log N_{\varrho(i)} \leq N_{\varrho(i)}$$

and thus for $\varrho = \varrho(i)$ and $C = J + J'$

$$H_R(s,\mathbf{y}) \leq \frac{CN_{\varrho(i)}}{h^{d-1}}. \tag{3.15}$$

All chips reaching $\mathscr{S}_{\varrho(i)+h}$ need to pass through $\mathscr{S}_{\varrho(i)}$ first, so by the Abelian property of quasirandom walks

$$N_{\varrho(i)+h} = \sum_{\mathbf{y}\in\mathscr{S}_{\varrho(i)+h}} H_R(s,\mathbf{y}).$$

Define $M_k = \#(A_n \cap \mathscr{S}_k)$. Then there are at most $M_{\varrho(i)+h}$ non-zero terms in the sum above, each one bounded by (3.15). Hence, by definition of $\varrho(i)$,

$$M_{\varrho(i)+h} \geq N_{\varrho(i)+h}\frac{h^{d-1}}{CN_{\varrho(i)}} \geq \frac{h^{d-1}}{2C}$$

and by summing over h

$$\sum_{h=0}^{\varrho(i+1)-\varrho(i)-1} M_{\varrho(i)+h} \geq \frac{1}{2dC}\left(\varrho(i+1) - \varrho(i) - 1\right)^d. \tag{3.16}$$

The left side is at most $N_{\varrho(i)}$, as all chips ending in $\left(\bigcup_{h=0}^{\varrho(i+1)-\varrho(i)-1} \mathscr{S}_{\varrho(i)+h}\right)$ must first pass through $\mathscr{S}_{\varrho(i)}$, thus

$$\varrho(i+1) - \varrho(i) \leq \left(2dCN_{\varrho(i)}\right)^{1/d} + 1 < N_{\varrho(i)}^{2/(2d-1)}$$

for all $N_{\varrho(i)} \geq C' > (2dC)^{2d-1}$ sufficiently large. We now have $N_{\varrho(i+1)} \leq N_{\varrho(i)}/2$ for these i by definition of $\varrho(i)$. Thus, $N_{\varrho(a\log r)} < C'$ for some large a.

Using the first part of the theorem, $B_{r-c\log r}$ is fully occupied, thus

$$\sum_{\varrho\geq r} M_\varrho \leq \omega_d r^d - \omega_d(r - c\log r)^d \leq cd\omega_d r^{d-1}\log r$$

and, using (3.16),

$$\frac{1}{2dC}\sum_{i=0}^{a\log r}(\varrho(i+1)-\varrho(i)-1)^d \le \sum_{i=0}^{a\log r}\sum_{h=0}^{\varrho(i+1)-\varrho(i)-1} M_{\varrho(i)+h} \le cd\omega_d r^{d-1}\log r.$$

By Jensen's inequality, the left sum is maximized for all $(\varrho(i+1)-\varrho(i)-1)$ being equal, i. e. $(\varrho(i+1)-\varrho(i)) \le C''r^{1-1/d}$ for some C'' large enough, and thus

$$\varrho(a\log r) = r + \sum_{i=0}^{a\log r}(\varrho(i+1)-\varrho(i)) \le r + aC''r^{1-1/d}\log r.$$

From $N_{\varrho(a\log r)} < C'$ it follows that $0 = N_{\varrho(a\log r)+C'} \ge N_{r+aC''r^{1-1/d}\log r+C'}$ and, by setting $c' = C' + aC''$, we have $N_{r\left(1+c'r^{-1/d}\log r\right)} = 0$. This finishes the proof for (3.3), because no particle ever reaches the shell $\mathscr{S}_{r\left(1+c'r^{-1/d}\log r\right)}$. □

This result is to some extent stronger than Theorem 3.2.1, because

$$\begin{aligned}
\mathscr{L}\left((\omega_d/n)^{1/d}A_n^\square \triangle B_1\right) &= \frac{\omega_d}{n}\mathscr{L}(A_n^\square \triangle B_r)\\
&\le \frac{\omega_d}{n}\mathscr{L}\left(B_{r(1+c'r^{-1/d}\log r)} - B_{r-c\log r}\right)\\
&= \frac{\omega_d^2}{n}\left((r(1+c'r^{-1/d}\log r))^d - (r-c\log r)^d\right)\\
&= \mathscr{O}(r^d/n\cdot r^{-1/d}\log r) = \mathscr{O}(n^{-1/d^2}\log n).
\end{aligned}$$

However, the speed of convergence for $\mathscr{L}\left((\omega_d/n)^{1/d}A_n^\square \triangle B_1\right)$ given in [LP08] is faster than what we get here. Although Theorem 3.2.4 already reduces the radius differences to $\mathscr{O}(r^{(d-1)/d}\log r) = \mathscr{O}(n^{(d-1)/d^2}\log n)$, many people expect them to be bounded by a constant like it is the case for the divisible sandpile presented in the next section.

This special result, however, exists for one-dimensional rotor-router aggregation. The proof was first given by Levine in [Lev02, Theorem 3.1]. For the two-dimensional model, very recent calculations by Friedrich and Levine in [FL10] show radius differences of less than two for up to ten billion chips. They also present an algorithm which allows fast calculation of the set A_n for large n.

Theorem 3.2.5 (Spherical bounds for quasirandom IDLA on $\mathbb{Z}^1$, [Lev02])
Let $(A_n)_{n\ge 1}$ be the sets generated by quasirandom IDLA on $\mathbb{Z}^1$ with arbitrary initial configuration. Then there exist constants c and c', such that for all $n \in \mathbb{N}$

$$B_{n/2-c} \subseteq A_n^\square \subseteq B_{n/2+c'}.$$

3.3. Linear growth models

In this section we want to study the *divisible sandpile* model, named after the similar *classical sandpile* model described in [Dha90, FR08]. The divisible sandpile was first studied in [LP09], though we want to present a slightly generalized version. The main idea is that, instead of routing an integer number of chips, one can split them and consequently distribute fractional amounts of chips evenly amongst the neighbors. Due to its similarity to quasirandom IDLA we will also name it *linear IDLA*, similar to the linear analogon of quasirandom walks described in Chapter 2. We will treat these similarities more extensively in the next chapter and now focus on basic properties of linear IDLA.

Definition 3.3.1 (Divisible sandpile / linear internal diffusion limited aggregation)
Let $\emptyset \neq \Theta_1, \Theta_2, \ldots \subseteq \mathbb{Z}^d$*, such that for all* $\mathbf{x} \in \mathbb{Z}^d$ *there are infinitely many* m *with* $\mathbf{x} \in \Theta_m$*. The sequence* $(\Theta_m)_{m\in\mathbb{N}}$ *is called* toppling sequence. *Let* $A^{\mathsf{L}}_{n,0} : \mathbb{Z}^d \to \mathbb{N}_{\geq 0}$, $A^{\mathsf{L}}_{n,0} = n\mathbf{1}_{\{0\}}$ *be the initial distribution for* $n \in \mathbb{N}_{\geq 0}$ *chips.*

For $m \in \mathbb{N}$*, let* $h^{\mathsf{L}}_m(\mathbf{x}) = \max\left(A^{\mathsf{L}}_{n,m}(\mathbf{x}) - 1, 0\right)$ *and define* $A^{\mathsf{L}}_{n,m} : \mathbb{Z}^d \to \mathbb{Q}_{\geq 0}$ *iteratively by*

$$A^{\mathsf{L}}_{n,m}(\mathbf{x}) = \min\left(A^{\mathsf{L}}_{n,m-1}(\mathbf{x}), 1\right) + \frac{1}{2d} \sum_{\mathbf{y}\in N(\mathbf{x})\cap\Theta_m} h^{\mathsf{L}}_{m-1}(\mathbf{y}) \qquad \textit{if } \mathbf{x} \in \Theta_m \textit{ and}$$

$$A^{\mathsf{L}}_{n,m}(\mathbf{x}) = A^{\mathsf{L}}_{n,m-1}(\mathbf{x}) + \frac{1}{2d} \sum_{\mathbf{y}\in N(\mathbf{x})\cap\Theta_m} h^{\mathsf{L}}_{m-1}(\mathbf{y}) \qquad \textit{otherwise.}$$

We say the sequence $\left(A^{\mathsf{L}}_{n,m}\right)_{m\in\mathbb{N}}$ *represents* linear internal diffusion limited aggregation *or the* divisible sandpile model *with* n *chips on* $\mathbb{Z}^d$*. For fixed* n*, we further define* $A^{\mathsf{L}}_n = A^{\mathsf{L}}_{n,\infty} = \lim_{m\to\infty} A^{\mathsf{L}}_{n,m}$ *to be the limit chip distribution.*

Remark. At this point, we only consider the case of n chips starting from the origin but it is an easy task to generalize the definition to arbitrary starting distributions $A^{\mathsf{L}}_{n,0}$. Note that we now identify $A^{\mathsf{L}}_{n,m}$ with functions, indicating the number of chips that actually lie on one field, instead of using sets as before.

Note that the limit $A^{\mathsf{L}}_{n,\infty}$ is not per se guaranteed to exist or to be well-defined for all n. This will be treated in Lemma 3.3.3 and Proposition 3.3.4. We also show that this limit is independent of the choice of the toppling sequence, which, to our knowledge, has not been shown before in such a general setting.

Example 3.3.2 (Divisible sandpile on $\mathbb{Z}^2$)
In Figure 3.5, we show the functions $A^{\mathsf{L}}_{n,m}$ and A^{L}_n for the linear IDLA model on $\mathbb{Z}^2$. We color fully occupied fields in black and partially occupied fields in gray shade. In the top row we have $n = 100$, in the bottom row $n = 1000$.

As this example illustrates, the divisible sandpile also has a highly circular shape. As we are going to see later on, the difference between these two models seems to be very low. This

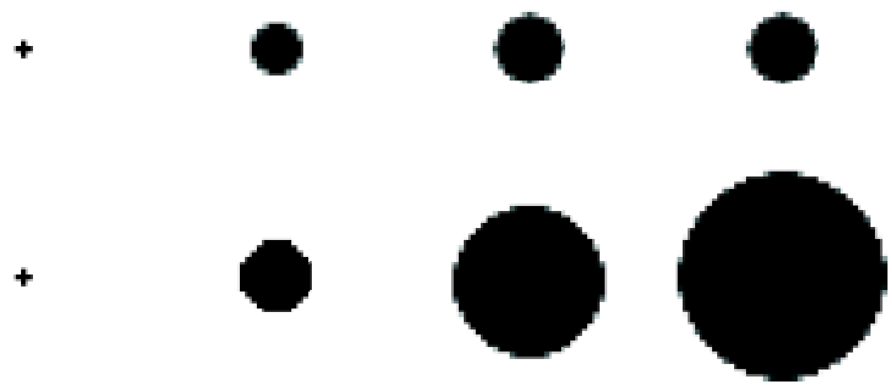


Figure 3.5.: The function $A^{\mathsf{L}}_{n,m}$ for the divisible sandpile on $\mathbb{Z}^2$ with $n \in \{100, 1000\}$, $m \in \{1, 10, 100, \infty\}$ and toppling sequence $\Theta_m = \mathbb{Z}^2$ for all m.

may possibly lead to some strategies for proving constant bounded radius differences also for quasirandom IDLA.

Lemma 3.3.3 (Limit distributions for single vertex toppling sequences, [LP09])
As in the setting above, let $A^{\mathsf{L}}_{n,m}$ represent linear IDLA on $\mathbb{Z}^d$, now with toppling sequence $\Theta_1, \Theta_2, \ldots \subseteq \mathbb{Z}^d$, such that $\Theta_m = \{\mathbf{x}_m\}$ for all $m \in \mathbb{N}$ and some $\mathbf{x}_m$. Again, we define the odometer function *u as the total amount of mass emitted from $\mathbf{x}$ by*

$$u_m(\mathbf{x}) = \sum_{\substack{k=1 \\ \mathbf{x}=\mathbf{x}_k}}^{m} A^{\mathsf{L}}_{n,k-1}(\mathbf{x}) - A^{\mathsf{L}}_{n,k}(\mathbf{x})$$

and $u(\mathbf{x}) = \lim_{m\to\infty} u_m(\mathbf{x})$. Then, $u = \lim_{m\to\infty} u_m$ exists and satisfies

$$A^{\mathsf{L}}_n = n\mathbf{1}_{\{0\}} + \Delta u \leq 1. \tag{3.17}$$

Proof. Let $B = B_R$ be a ball centered at the origin containing all points within L^1-distance n from 0. Obviously, $A^{\mathsf{L}}_{n,m}$ is supported in B. Define the *quadratic weight*

$$Q_m = \sum_{\mathbf{x}\in\mathbb{Z}^d} A^{\mathsf{L}}_{n,m}(\mathbf{x})\|\mathbf{x}\|^2 \leq nR^2.$$

Using that $A^{\mathsf{L}}_{n,m}(\mathbf{y}) = A^{\mathsf{L}}_{n,m-1}(\mathbf{y})$ if $\mathbf{y}$ is not adjacent to the vertex toppled and that $A^{\mathsf{L}}_{n,m}(\mathbf{y}) - A^{\mathsf{L}}_{n,m-1}(\mathbf{y}) = \frac{1}{2d}\left(A^{\mathsf{L}}_{n,m-1}(\mathbf{x}_m) - A^{\mathsf{L}}_{n,m}(\mathbf{x}_m)\right)$ for $\mathbf{y} \in N(\mathbf{x}_m)$, we have

$$Q_m - Q_{m-1} = \left(A^{\mathsf{L}}_{n,m-1}(\mathbf{x}_m) - A^{\mathsf{L}}_{n,m}(\mathbf{x}_m)\right)\left(\frac{1}{2d}\sum_{\mathbf{y}\in N(\mathbf{x}_m)} \|\mathbf{y}\|^2 - \|\mathbf{x}_m\|^2\right)$$

$$= A^{\mathsf{L}}_{n,m-1}(\mathbf{x}_m) - A^{\mathsf{L}}_{n,m}(\mathbf{x}_m)$$

and by summing over m

$$Q_m - Q_1 = \sum_{\mathbf{x}\in\mathbb{Z}^d} u_m(\mathbf{x}).$$

For fixed $\mathbf{x}$, the sequence $(u_m(\mathbf{x}))_{m\in\mathbb{N}}$ is bounded and also increasing by definition, hence convergent. Given any two neighboring vertices $\mathbf{x}$ and $\mathbf{y}$, it is clear that $\mathbf{y}$ emits $u_m(\mathbf{y})/2d$ chips to $\mathbf{x}$ up to time m. Thus, $\mathbf{x}$ receives mass $\frac{1}{2d}\sum_{\mathbf{y}\in N(\mathbf{x})} u_m(\mathbf{y})$ from its neighbors up to time m, yielding

$$A^{\mathsf{L}}_{n,m}(\mathbf{x}) = A^{\mathsf{L}}_{n,0}(\mathbf{x}) + \Delta u_m(\mathbf{x}) = n\mathbf{1}_{\{0\}}(\mathbf{x}) + \Delta u_m(\mathbf{x}).$$

Using $u_m \to u$, we have $A^{\mathsf{L}}_{n,m} \to A^{\mathsf{L}}_n = A^{\mathsf{L}}_{n,0} + \Delta u$. As $A^{\mathsf{L}}_{n,m}(\mathbf{x}_m) \le 1$ for all m, also $A^{\mathsf{L}}_n(\mathbf{x}) \le 1$ holds for all $\mathbf{x}$. □

Proposition 3.3.4 (Uniqueness of limit distributions)
As in the setting above, let $A^{\mathsf{L}}_{n,m}$ represent linear IDLA on $\mathbb{Z}^d$ with arbitrary toppling sequence $\Theta_1, \Theta_2, \ldots \subseteq \mathbb{Z}^d$. Then $A^{\mathsf{L}}_n = A^{\mathsf{L}}_{n,\infty}$ exists and is independent of the choice of Θ_m.

Proof. Fix $n \in \mathbb{N}_{\ge 0}$. From Lemma 3.3.3 we know that $\lim_{m\to\infty} A^{\mathsf{L}}_{n,m}$ exists for an arbitrary toppling sequence $\Theta_1, \Theta_2, \ldots \subseteq \mathbb{Z}^d$ of the form $\Theta_m = \{\mathbf{x}_m\}$. Let $\hat{\Theta}_1, \hat{\Theta}_2, \ldots \subseteq \mathbb{Z}^d$ be any other toppling sequence resulting in functions $\hat{A}^{\mathsf{L}}_{n,m}$. We define $\|f\|_1 = \sum_{\mathbf{x}\in\mathbb{Z}^d} |f(\mathbf{x})|$ for some $f : \mathbb{Z}^d \to \mathbb{R}$ with finite support.

Let $\varepsilon > 0$ arbitrary and let $m_1 \ge 1$ be such that

$$\|A^{\mathsf{L}}_{n,m} - A^{\mathsf{L}}_n\|_1 < \varepsilon/2 \quad \text{for all } m \ge m_1. \tag{3.18}$$

For any $\mathbf{x}$, we know that $A^{\mathsf{L}}_{n,m_1}(\mathbf{x}) \ge 1$ implies $A^{\mathsf{L}}_n(\mathbf{x}) = 1$, so we can estimate the amount of "unspread" mass at time m_1 by

$$\|\mathbf{1}_{\{A^{\mathsf{L}}_{n,m_1}\ge 1\}\cup\{A^{\mathsf{L}}_n<1\}}(A^{\mathsf{L}}_{n,m_1} - A^{\mathsf{L}}_n)\|_1 < \varepsilon/4. \tag{3.19}$$

Now, choose $D \in \mathbb{N}$ such that $(2d)^{D-1}A^{\mathsf{L}}_{n,m}(\mathbf{x}) \in \mathbb{Z}$ for all $\mathbf{x}$ and $m \le m_1$. Then, instead of having n particles with mass 1 on the origin, we can also say we have $(2d)^D n$ particles with mass $(2d)^{-D}$ starting from the origin, performing classical rotor-router aggregation (with up to $(2d)^D$ chips on one field), as none of them will have to split. Label these particles from 1 to $(2d)^D n$. For any i and up to time m_1, particle i will move along the path $P^i = (P^i_0, \ldots, P^i_{k(i)})$ of length $k(i)$ for $P^i_0 = 0$ and $P^i_{k(i)}$ being its final position at time m_1.

Note that particle i only moves if the vertex on which it currently lies gets toppled and that it does so independently of the order in which the other vertices get toppled. For $0 \le l < k(i)$, let ϑ^i_l be the time when vertex P^i_l gets toppled (i. e. $P^i_l \in \Theta_{\vartheta^i_l}$) so that particle i reaches P^i_{l+1}.

In particular, ϑ_l^i is the first time that P_l^i gets toppled after ϑ_{l-1}^i, as particles remain on their position if and only if they have reached their final position.

Similarly, let $\hat{\vartheta}_0^i$ be the first time 0 gets toppled in $\hat{\Theta}_m$, i. e. $0 \in \hat{\Theta}_{\hat{\vartheta}_0^i}$. Then, for $1 \leq l < k(i)$, define $\hat{\vartheta}_l^i$ to be the first time after $\hat{\vartheta}_{l-1}^i$ such that $P_l^i \in \hat{\Theta}_{\hat{\vartheta}_l^i}$. Choose

$$m_2 = \max_{1 \leq i \leq (2d)^D n} \{\hat{\vartheta}_{k(i)-1}^i\}.$$

Then, looking at $\hat{A}_{n,m_2}^{\mathsf{L}}$, all particles have at least reached their final position of A_{n,m_1}^{L}, though they may have been split further. However, all particles which at any time reach their final position of A_n^{L} will remain there. Thus, only mass that was unspread in A_{n,m_1}^{L} could have been moved further in $\hat{A}_{n,m}^{\mathsf{L}}$ for some $m \geq m_2$. Therefore, by (3.19),

$$\|\hat{A}_{n,m}^{\mathsf{L}} - A_{n,m_1}^{\mathsf{L}}\|_1 < 2\varepsilon/4 \qquad \text{for all } m \geq m_2.$$

Together with (3.18), we get

$$\|\hat{A}_{n,m}^{\mathsf{L}} - A_n^{\mathsf{L}}\|_1 < \varepsilon \qquad \text{for all } m \geq m_2, \qquad \square$$

which means that $\lim_{m\to\infty} \hat{A}_{n,m}^{\mathsf{L}} = A_n^{\mathsf{L}}$.

Finally, we are able to prove a much better result concerning circularity of the resulting occupied sites A_n^{L} for linear IDLA. This is certainly expected as the regular structure of the linear model makes analysis much easier.

Theorem 3.3.5 (Spherical bounds for linear IDLA, [LP09])
Let $A_{n,m}^{\mathsf{L}}$ represent linear IDLA for fixed n and m and toppling sequence $\Theta_1, \Theta_2, \ldots \subseteq \mathbb{Z}^d$ and let $D_n = \{\mathbf{x} \in \mathbb{Z}^d : A_n^{\mathsf{L}}(\mathbf{x}) = 1\}$. Then there exist constants c and c′ depending only on d, such that

$$B_{r-c} \subseteq D_n^{\square} \subseteq B_{r+c'}, \tag{3.20}$$

where $r = (n/\omega_d)^{1/d}$.

Proof. We assume $d \geq 2$, as the case $d = 1$ is obvious. We say a function $s : \mathbb{Z}^d \to \mathbb{R}$ is *superharmonic* if $\Delta s \leq 0$. Superharmonic functions have been studied intensively in connection with PDEs. Given any function γ, the *least superharmonic majorant* of γ is defined as $s = \inf\{f : f \text{ is superharmonic and } f \geq \gamma\}$ and one can show that s itself is superharmonic. Fix $n > 0$ and let

$$\tilde{\gamma}_d(\mathbf{x}) = \|\mathbf{x}\|^2 + n g(\mathbf{x})$$

as previously defined, where g shall denote the Green's function given in (3.4).

We know from Proposition 3.3.4 that the exact choice of toppling sequence has no influence on A_n. Thus, by Lemma 3.3.3, $\Delta u - \Delta\tilde{\gamma}_d = A_n^{\mathsf{L}} - n\mathbf{1}_{\{0\}} - 1 + n\mathbf{1}_{\{0\}} \leq 0$, so the difference $u - \tilde{\gamma}_d$

is superharmonic. Let $\gamma_d = \tilde{\gamma}_d + C_0$ be defined as in (3.5). By (3.7) and $u \geq 0$, we have $u - \gamma_d \geq -C_1$ on $(B_{r+1} - B_{r-1})$ for C_1 depending only on d. Since $u - \gamma_d$ is superharmonic, its minimum in the ball B_r is attained at the boundary. Using (3.6), we obtain

$$u(\mathbf{x}) \geq \gamma_d(\mathbf{x}) - C_1 \geq (r - \|\mathbf{x}\|)^2 - C_2 r^d / \|\mathbf{x}\|^d \tag{3.21}$$

for $\mathbf{x} \in B_r$ and a constant C_2. Thus, there exists C_3 such that $u(\mathbf{x}) > 0$ whenever $r/3 \leq \|\mathbf{x}\| < r - C_3$. By (3.8), we get for $\|\mathbf{x}\| < r/3$ and $r > 2\sqrt{C_1}$ that $u(\mathbf{x}) \geq r^2/4 - C_1 > 0$. Let $c = \max(C_3, 2\sqrt{C_1})$, then $u(\mathbf{x}) > 0$ on B_{r-c} and $B_{r-c} \subseteq D_n^{\square}$.

Further, note that $u - \gamma_d$ is harmonic on D_n. From (3.6) and (3.8) we know that there exists $a > 0$, such that $\gamma_d \geq -a$ everywhere. Since $u = 0$ on D_n^c, we have that $u - \gamma_d \leq a$ on D_m. For $\mathbf{x} \in D_n$ with $r - 1 < \|\mathbf{x}\| \leq r$, we have by (3.7)

$$u(\mathbf{x}) \leq \gamma_d(\mathbf{x}) + a \leq C_4 \tag{3.22}$$

with some constant C_4. Let $\hat{\mathbf{x}}_0$ be one point farthest away from the origin in D_n. For $i \geq 0$ and $\hat{\mathbf{x}}_i$ given, let $\hat{\mathbf{x}}_{i+1}$ be the neighbor of $\hat{\mathbf{x}}_i$ maximizing $u(\hat{\mathbf{x}}_{i+1})$. Then

$$\begin{aligned} u(\hat{\mathbf{x}}_{i+1}) - 1 &\geq \frac{1}{2d} \sum_{\mathbf{y} \in N(\hat{\mathbf{x}}_i)} u(\mathbf{y}) - 1 \\ &= u(\hat{\mathbf{x}}_i) + \Delta u(\hat{\mathbf{x}}_i) - 1 \\ &= u(\hat{\mathbf{x}}_i), \end{aligned}$$

using that $\Delta u = 1$ in D_n. Together with (3.22) and $u(\hat{\mathbf{x}}_0) \geq 0$, this means that $\|\hat{\mathbf{x}}_0\| \leq r + C_4$. Thus, $D_n^{\square} \subseteq B_{r+C_4+1} = B_{r+c'}$. □

The last theorem is a strong result on the circularity of the divisible sandpile. However, the equivalent has not yet been proven for quasirandom IDLA. In the next chapter, we give a short outlook on possible strategies for proving constant bounded radius difference and present some general enhancements of the quasirandom walk.

4. Outlook

4.1. Quasirandom Markov chains

The main focus of this chapter lies on possible extensions to the field of quasirandom walks, from which future work in that area could arise. First, we want to give a quick overview on a possible extension of the quasirandom walk as suggested in [HP10], which can be used to simulate general Markov chains with countable state space.

The main difference to the rotor model as defined in Chapter 2 is that, until now, we were only able to simulate Markov chains with equal transition probabilities for each neighbor if we did not want to resort to using multigraphs. A generalization for rational transition probabilities is possible if we change the relative number of occurrences for each rotor position during one rotation according to its transition probability.

Definition 4.1.1 (Quasirandom Markov chain)
Let $G = (V, E)$ be a countable graph. As it was the case for quasirandom walks, let $x_0 \in V$ and let $\sigma = \sigma_0 : V \to \mathbb{Z}_{d(.)}$ be the initial rotor state.

Given a Markov chain Y on G with rational transition probabilities $p : V \times V \to [0,1]$, where $p(x,y) = 0$ for x and y not adjacent, associate a fixed integer $\delta(x)$ to each vertex x and let $n_x : \mathbb{Z}_{\delta(x)} \to N(x) \subseteq V$, such that

$$p(x,y) = \frac{\#\{t \in \mathbb{Z}_{\delta(x)} : n_x(t) = y\}}{\delta(x)} \quad \text{for all } x, y \in V.$$

Similar to quasirandom walks, we say that the V-valued sequence $X = (X_0, X_1, X_2, \ldots)$ is a quasirandom Markov chain *to Y starting in x_0 if*

$$\begin{aligned} X_0 &= x_0 \\ X_{t+1} &= n_{X_t}\left(\sigma_t(X_t)\right) & \forall t \in \mathbb{N}_{\geq 0} \end{aligned}$$

where

$$\sigma_{t+1}(x) = \begin{cases} \sigma_t(x) + 1 \pmod{\delta(x)} & x = X_t \\ \sigma_t(x) & x \neq X_t \end{cases} \qquad \forall t \in \mathbb{N}_{\geq 0}, x \in V.$$

One can easily see that this definition is only a slight alteration of quasirandom walks. Thus, we expect similar properties also in comparison to their random counterparts. For a (quasirandom) Markov chain X, let $n_t^X(x) = \#\{0 \le s \le t-1 : X_s = x\}$ be the number of visits to x before time t and let $\tau_x^X = \min\{t \ge 0 : X_t = x\}$ the first hitting time of x. The following result shows the similarity of those properties between quasirandom and random Markov chains.

Theorem 4.1.2 (Hitting probabilities and hitting times, [HP10])
Given a Markov chain Y on $G = (V, E)$ with transition probabilities p, let X be an arbitrary quasirandom Markov chain to Y as previously defined. Fix $v \neq w \in V$ and let $h(x) = \mathbb{P}_x(\tau_v^Y < \tau_w^Y)$ the hitting probability of v. If

$$K_1 = 1 + \frac{1}{2} \sum_{\substack{x \in V - \{v,w\} \\ y \in V}} \delta(x) p(x,y) |h(x) - h(y)|$$

is finite then for all $t \ge 0$ and $x \notin \{v, w\}$

$$\left| h(x) - \frac{n_t^X(v)}{n_t^X(v) + n_t^X(w)} \right| \le \frac{K_1}{n_t^X(v) + n_t^X(w)}.$$

For fixed $v \in V$, let $k(x) = \mathbb{E}_x[\tau_v^Y]$ be the expected hitting time of v. If, in addition, V is finite, then set

$$K_2 = \max_{y \in V} k(y) + \frac{1}{2} \sum_{\substack{x \in V - \{v\} \\ y \in V}} \delta(x) p(x,y) |k(x) - k(y) - 1|$$

and we have

$$\left| k(x) + 1 - \frac{t}{n_t^X(v)} \right| \le \frac{K_2}{n_t^X(v)}.$$

This implies that the proportion of the number of visits to v and w converges to the hitting probability $h(x)$ of the Markov chain if v and w are visited infinitely often. Also, after n visits to v or w, the difference from that limit is at most K/n for a constant K, whereas it is of the order $1/\sqrt{n}$ for classical Markov chains. The second part shows that the average time needed to hit v from x is also similar to the expectation of the Markov chain.

4.2. Hybrid aggregation models

In Section 3.3 we saw how extraordinarily circular the divisible sandpile was. However, instead of defining this model independently, we also suggested a connection to quasirandom IDLA similar to the one of the linear model to quasirandom walks in Sections 2.3 and 2.4. There, all results indicate a strong relationship between those two, which could possibly be extended to their aggregating versions.

So, in our new perspective of linear IDLA, n chips start from the origin and then perform the *linear walk*, meaning that all chips but one get distributed evenly amongst all neighbors. Note that in Chapter 2 the linear walk was equal to the expectation of classical random walks, which is not the case now. To simplify comparison, we give an equivalent definition of quasirandom IDLA by initially putting n chips on the origin and using similar functions $A^{\mathsf{Q}}_{n,m}$ and toppling sequences as in Definition 3.3.1.

Proposition 4.2.1 (Quasirandom IDLA)
Let $\Theta_1, \Theta_2, \ldots \subseteq \mathbb{Z}^d$ be a toppling sequence and $\sigma_0 : \mathbb{Z}^d \to \mathbb{Z}_{2d}$ an arbitrary initial rotor state. Set $A^{\mathsf{Q}}_{n,0}(\mathbf{x}) = n\mathbf{1}_{\{0\}}(\mathbf{x})$ to be the initial distribution for $n \in \mathbb{N}_{\geq 0}$ chips. For $m \in \mathbb{N}_{\geq 0}$, let $h^{\mathsf{Q}}_m(\mathbf{x}) = \max\left(A^{\mathsf{Q}}_{n,m}(\mathbf{x}) - 1, 0\right)$ and let $N_m(\mathbf{x}) \subseteq N(\mathbf{x})$ be the set of neighbors of $\mathbf{x}$ at which the rotors on $\mathbf{x}$ in one of the positions $\left\{\sigma_m(\mathbf{x}), \sigma_m(\mathbf{x})+1, \ldots, \sigma_m(\mathbf{x}) + \left(h^{\mathsf{Q}}_{m-1}(\mathbf{x}) - 1 \mod 2d\right)\right\}$ would be pointing.

Then, for $m \geq 1$, we define $A^{\mathsf{Q}}_{n,m} : \mathbb{Z}^d \to \mathbb{N}_{\geq 0}$ iteratively by

$$A^{\mathsf{Q}}_{n,m}(\mathbf{x}) = \min\left(A^{\mathsf{Q}}_{n,m-1}(\mathbf{x}), 1\right) + \left(\sum_{\mathbf{y} \in N(\mathbf{x}) \cap \Theta_m} \left\lfloor \frac{1}{2d} h^{\mathsf{Q}}_{m-1}(\mathbf{y}) \right\rfloor + \mathbf{1}_{N_m(\mathbf{y})}(\mathbf{x})\right) \quad \text{if } \mathbf{x} \in \Theta_m \text{ and}$$

$$A^{\mathsf{Q}}_{n,m}(\mathbf{x}) = A^{\mathsf{Q}}_{n,m-1}(\mathbf{x}) + \left(\sum_{\mathbf{y} \in N(\mathbf{x}) \cap \Theta_m} \left\lfloor \frac{1}{2d} h^{\mathsf{Q}}_{m-1}(\mathbf{y}) \right\rfloor + \mathbf{1}_{N_m(\mathbf{y})}(\mathbf{x})\right) \quad \text{otherwise,}$$

where

$$\sigma_m(\mathbf{x}) = \begin{cases} \sigma_{m-1}(\mathbf{x}) + h^{\mathsf{Q}}_{m-1}(\mathbf{x}) \pmod{2d} & \text{if } \mathbf{x} \in \Theta_m \\ \sigma_{m-1}(\mathbf{x}) & \text{otherwise.} \end{cases}$$

For fixed n, we define $A^{\mathsf{Q}}_n = A^{\mathsf{Q}}_{n,\infty} = \lim_{m\to\infty} A^{\mathsf{Q}}_{n,m}$ to be the limit chip distribution. This limit exists independently of the choice of the toppling sequence and satisfies

$$A^{\mathsf{Q}}_n = \mathbf{1}_{A_n},$$

where A_n is the set created from quasirandom IDLA with equal initial rotor configuration σ_0 as in Definition 3.1.3.

Proof. This is an immediate consequence of the Abelian property (c.f. Section 3.1) for quasirandom walks. □

We can now directly compare the functions $A^{\mathsf{L}}_{n,m}$ and $A^{\mathsf{Q}}_{n,m}$ for linear and quasirandom IDLA as done in Figure 4.1. In this example, we let $n = 1000$ and show the two functions for different values of m (c.f. Figure 3.5). On the bottom, we plot the absolute difference $\left|A^{\mathsf{L}}_{n,m} - A^{\mathsf{Q}}_{n,m}\right|$, which seems to be very small in our calculation and, especially, is almost everywhere zero in the limit distribution. Again, we color values greater or equal 1 in black and function values in $(0,1)$ in gray shade.

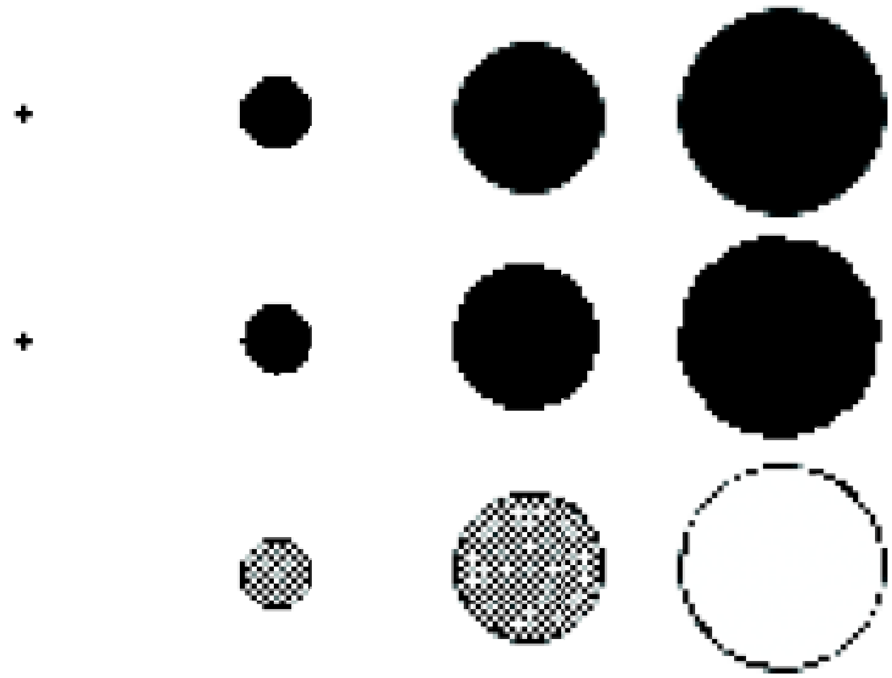


Figure 4.1.: The functions $A^{\mathrm{L}}_{n,m}$, $A^{\mathrm{Q}}_{n,m}$, and $\left|A^{\mathrm{L}}_{n,m} - A^{\mathrm{Q}}_{n,m}\right|$, where $n = 1000$, $m \in \{1, 10, 100, \infty\}$ and constant toppling sequence $\Theta_m = \mathbb{Z}^2$ for all m. The initial rotor state is $\rightarrow$ for all fields with rotor sequence $(\rightarrow, \downarrow, \leftarrow, \uparrow)$.

Note that in Sections 2.3 and 2.4 we used the hybrid model represented by $E(x, t_1, t_2)$ to approximate differences between linear and quasirandom walks. With our notation from above, this can be expanded to the aggregation models. Define $A^{(m_1)}_{n,m_2} : \mathbb{Z}^d \rightarrow \mathbb{R} \geq 0$ for $0 \leq m_1 \leq m_2$ to be the function which results from letting n chips start at the origin, toppling the whole structure m_1 times using rotor-directions, and then toppling it $m_2 - m_1$ times as in the divisible sandpile. We will call $A^{(m_1)}_{n,m_2}$ the *hybrid* IDLA model.

Again, we set the limit $A^{(m)}_n = \lim_{k\rightarrow\infty} A^{(m)}_{n,k}$ for $m \in \mathbb{N}_{\geq 0}$. This leads to the interesting sequence $\left(A^{(m)}_n\right)_{m\in\mathbb{N}_{\geq 0}}$, for which

$$A^{(0)}_n = A^{\mathrm{L}}_n \text{ and } \lim_{m\rightarrow\infty} A^{(m)}_n = A^{\mathrm{Q}}_n.$$

Note that the sequence always converges, as for fixed n and for every toppling sequence there exists a $K > 0$, such that $A^{\mathrm{Q}}_{n,k} = A^{\mathrm{Q}}_{n,K}$ for all $k > K$. Figure 4.2 shows $A^{(m)}_n$ for $n = 100$ and $m \in \{0, 50, 55, \infty\}$. Although we have not yet been able to state rigorous results for the differences $\left|A^{(m)}_n - A^{(m-1)}_n\right|$, we still think that this could be a promising way for further investigations.

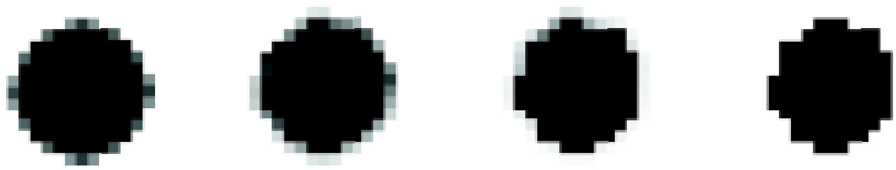


Figure 4.2.: The functions $A_n^{(m)}$ for $n = 100$ and $m \in \{0, 50, 55, \infty\}$. The initial rotor state is $\rightarrow$ for all fields with rotor sequence $(\rightarrow, \downarrow, \leftarrow, \uparrow)$.

4.3. Different underlying grids

Until now we have only considered graphs of the form $\mathbb{Z}^d$ as a basis for quasirandom walks and IDLA. However, it may be worth to study the behavior of these two models on different infinite graphs with a similar structure. In particular, we ask the question whether it would be possible that the circular structure seen in $\mathbb{Z}^d$ also evolves on different graphs G embedded in $\mathbb{R}^d$.

Definition 4.3.1 (Grids)
Let $d \in \mathbb{N}$. A countable graph G with vertex set $V \subseteq \mathbb{R}^d$ is called a grid *if there exists a $D > 0$, such that for all $\mathbf{y} \in \mathbb{R}^d$ there is a $\mathbf{x} \in V$ such that $\|\mathbf{y} - \mathbf{x}\| < D$.*

Obviously, $\mathbb{Z}^d$ is a grid for all $d \in \mathbb{N}$. The first thing one notices is that, for circularity to evolve from quasirandom IDLA on a grid, it seems that the grid must be regular to some extent. This means that for a classical random walk X on the grid the expectancy $\mathbb{E}_\mathbf{x}[X_1 - X_0]$ should be zero on average, or, even better, that $\mathbb{E}_\mathbf{x}[X_1 - X_0] = 0$ for all vertices $\mathbf{x}$. All simulations conducted on underlying grids where this was not the case, e. g. $\mathbb{Z}^d$ with edges left out, produced obviously non-circular results.

Definition 4.3.2 (Unbiased Grids)
We say a grid $G \subseteq \mathbb{R}^d$ is unbiased *if $\sum_{\mathbf{y} \in N_G(\mathbf{x})} (\mathbf{y} - \mathbf{x}) = 0$ for all vertices $\mathbf{x}$.*

Note that the condition mentioned above, that for a random walk $\mathbb{E}_\mathbf{x}[X_1 - X_0] = 0$ for all $\mathbf{x}$, is equivalent to our definition of unbiased grids. The classically used $\mathbb{Z}^d$ clearly is unbiased.

At this point, we want to give a few examples of unbiased grids on $\mathbb{Z}^2$. We have simulated quasirandom IDLA for those grids in Appendix A, where we plot the resulting sets A_n. It has proven to be useful to color areas instead of vertices. Thus, we associate a surrounding area with each vertex, which will be colored together with its according vertex.

The pretty self explanatory structure of our examples is presented in Figure 4.3 and we will call them the diagonal, triangle, hexagon, and octagon-square grid. The black circles and lines represent vertices and edges, whereas the gray lines indicate the border of the associated areas.

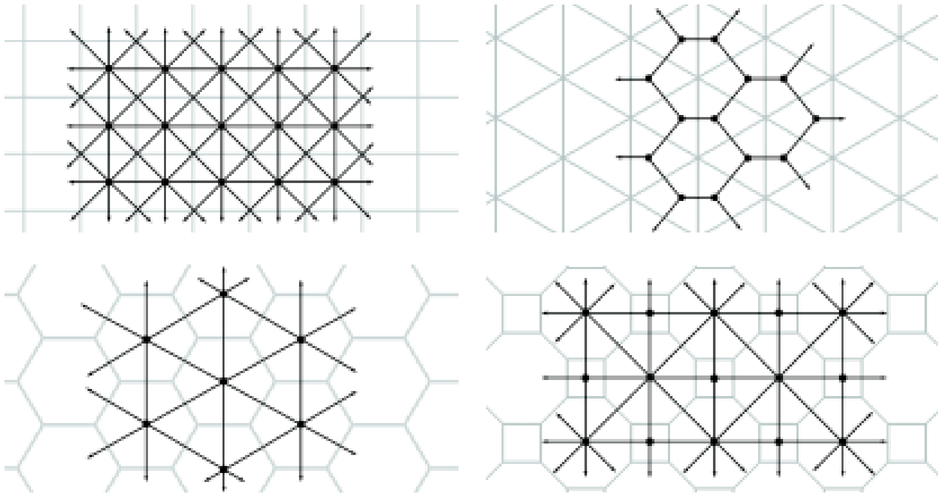


Figure 4.3.: An overview of some unbiased grids. Top left: diagonal grid, top right: triangle grid, bottom left: hexagon grid, and bottom right: octagon-square grid.

However, opposite to what one could think, not all unbiased grids seem to produce circles as the following example is going to demonstrate. We define the *harmonic grid* $G_H \subseteq \mathbb{Z}^2$ to be the following (multi-)graph $G_H = (V_H, E_H)$, where

$$V_H = \left\{\dots, -1-\frac{1}{2}-\frac{1}{3}, -1-\frac{1}{2}, -1, 0, 1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \dots\right\}^2$$

and two vertices $\mathbf{x}$ and $\mathbf{y}$ are adjacent if and only if for some $1 \le i \le d$ and $n \in \mathbb{N}$ we have that $(\mathbf{x}-\mathbf{y})_i \in \{-1/n, 1/n\}$ and $(\mathbf{x}-\mathbf{y})_j = 0$ for $j \neq i$. In this case, there shall be n edges between those vertices.

Figure 4.4 shows the center part of the harmonic grid. Note that, because the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, the harmonic grid is a grid as in Definition 4.3.1. The harmonic grid is also unbiased, as for all $\mathbf{x}$ the sum of the differences $\mathbf{x}-\mathbf{y}$ over all neighbors $\mathbf{y}$ is zero, if we account for the number of edges between the vertices. As shown in Figure A.10 in the appendix, the resulting set of occupied vertices after 1,000,000 chips is not circular but square-shaped. However, looking at the radius differences $|R_n - r_n|$ we can see that they stay relatively small even for large n, which is mainly due to the fact that the area of occupied vertices itself is only growing very slowly. In particular, we still think that this difference is at a maximum of order $\mathcal{O}(n^{(d-1)/d^2} \log n)$. We thus propose the following conjecture, similar to Theorem 3.2.4.

Conjecture 4.3.3 (Spherical bounds for quasirandom IDLA on unbiased grids)
Let $(A_n)_{n\geq 1}$ be the sets created by quasirandom IDLA on an unbiased grid $G \subseteq \mathbb{R}^d$. Let R_n and r_n be the outer and inner radius of A_n. Then

$$|R_n - r_n| = \mathcal{O}(n^{(d-1)/d^2} \log n)$$

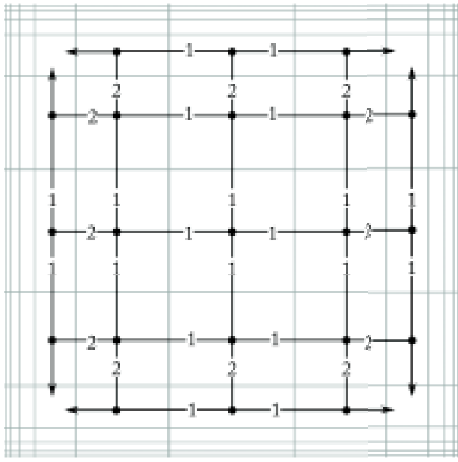


Figure 4.4.: The center part of the harmonic grid. Vertices and edges are in black, the areas for coloring are marked in gray. The numbers indicate the number of edges between the vertices.

or even, if some additional conditions hold,

$$|R_n - r_n| = \mathcal{O}(1).$$

Note that the first part is proven in Theorem 3.2.4 for the grid $\mathbb{Z}^d$. On the other hand, we are not sure whether the second part is true for the harmonic grid. The resulting square will possibly grow slowly but infinitely and thus result in an unbouded radius difference. It is important to see that the simple fact of a grid having multiple edges between vertices *not* necessarily implies supposedly non-constant radius differences. E. g. our calculations on the grid $\mathbb{Z}^2$ with two edges between each pair of neighbors indicate almost as small radius differences as for the normal $\mathbb{Z}^2$ grid. So, we suggest the following two "additional conditions" that could be used for Conjecture 4.3.3.

1. The supremum of the degrees of all vertices is finite.
2. The infimum of all distances between vertices is greater than zero.

Both conditions do not hold for the harmonic grid, but do so for all other grids shown in this section. Which (if any) of these conditions is the right choice, we do not know currently and leave for future research.

A. Experimental results and figures

On the following pages, we want to give a few experimental results that we obtained during our calculations. Especially, we show pictures of the sets A_n that are created by quasirandom IDLA for different grids and rotor directions. Usually, we use different colors for all occupied fields, depending on their rotor direction as suggested by Propp. The patterns that hereby arise are remarkable, although no satisfying explanation for their occurrence has been found, yet.

In addition, we present graphs showing the radius difference $|R_n - r_n|$ as defined in Definition 3.1.5 for large values of n. Besides different underlying grids and rotor directions, we will also give examples of quasirandom IDLA on $\mathbb{Z}^d$ for $d > 2$. Until now, all calculations suggest constant radius difference for the quasirandom aggregation model. To further interested readers, we would like to suggest a visit to James Propp's website at [Pro06], which gives an overview on the whole topic of quasirandom IDLA. He also lists an easy to use Java applet for self experimentation.

To create the following pictures, we used an own implementation of the rotor-router model, executed on a standard home computer. The radius difference graphs were created in Matlab. Both parts, the rotor-router and the user interface are programmed in Java, where each different rotor model was implemented as a subclass of an *abstract rotor* class. All classes are self-supporting, meaning that one can use them in connection with arbitrary interfaces. A full version of the rotor software, including an executable .jar file with many configuration options, can be obtained from the author via e-mail or from http://www-m9.ma.tum.de/Allgemeines/CarlGeorgHeise.

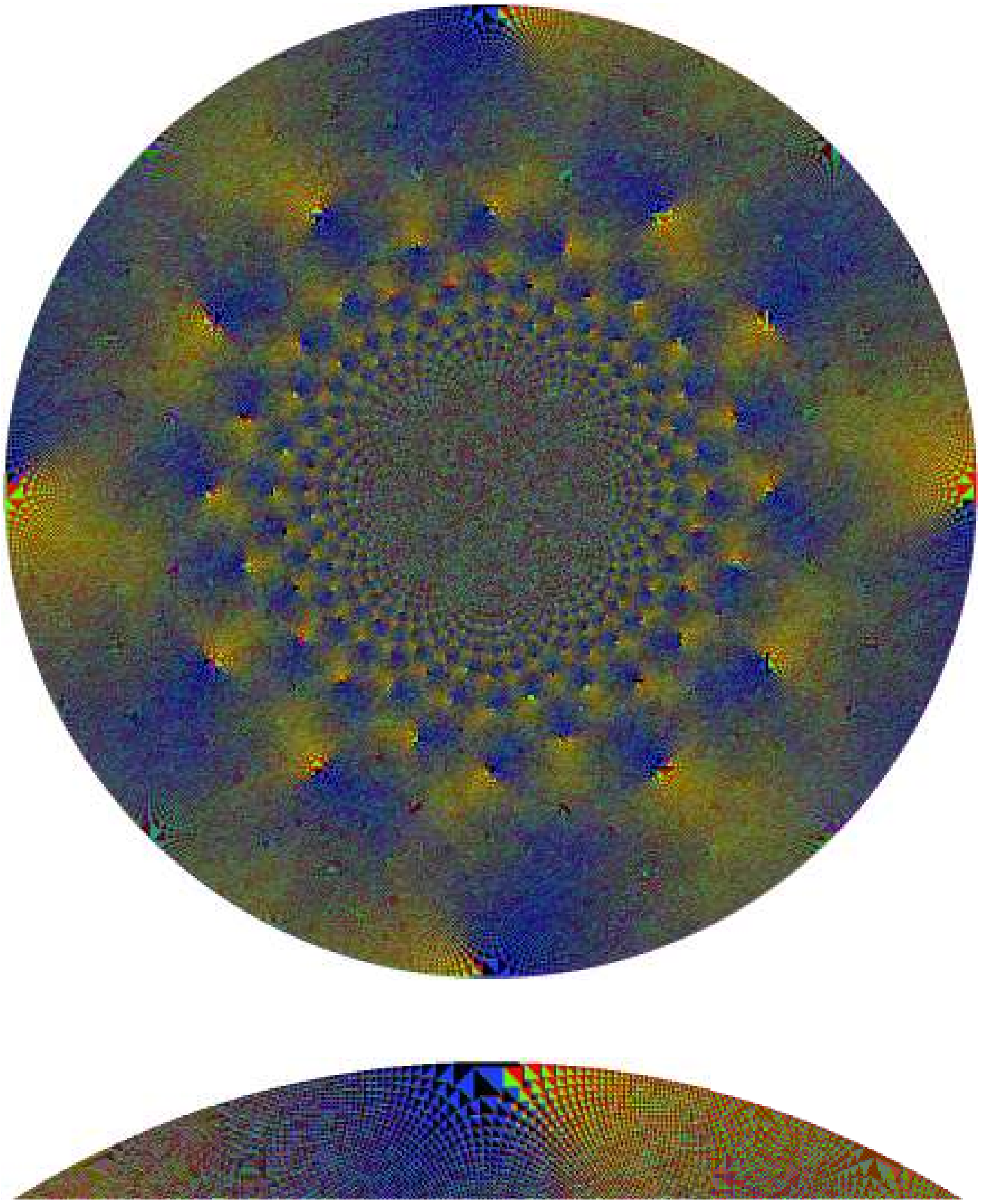


Figure A.1.: The set $A_{3,000,000}$ for quasirandom IDLA on $\mathbb{Z}^2$ with initial rotor state $\rightarrow$ for all fields and rotor sequence $(\rightarrow, \downarrow, \leftarrow, \uparrow)$. The fields are colored according to their rotor direction green, blue, red, and black respectively. The lower figure gives a detailed view of the top of the circle.

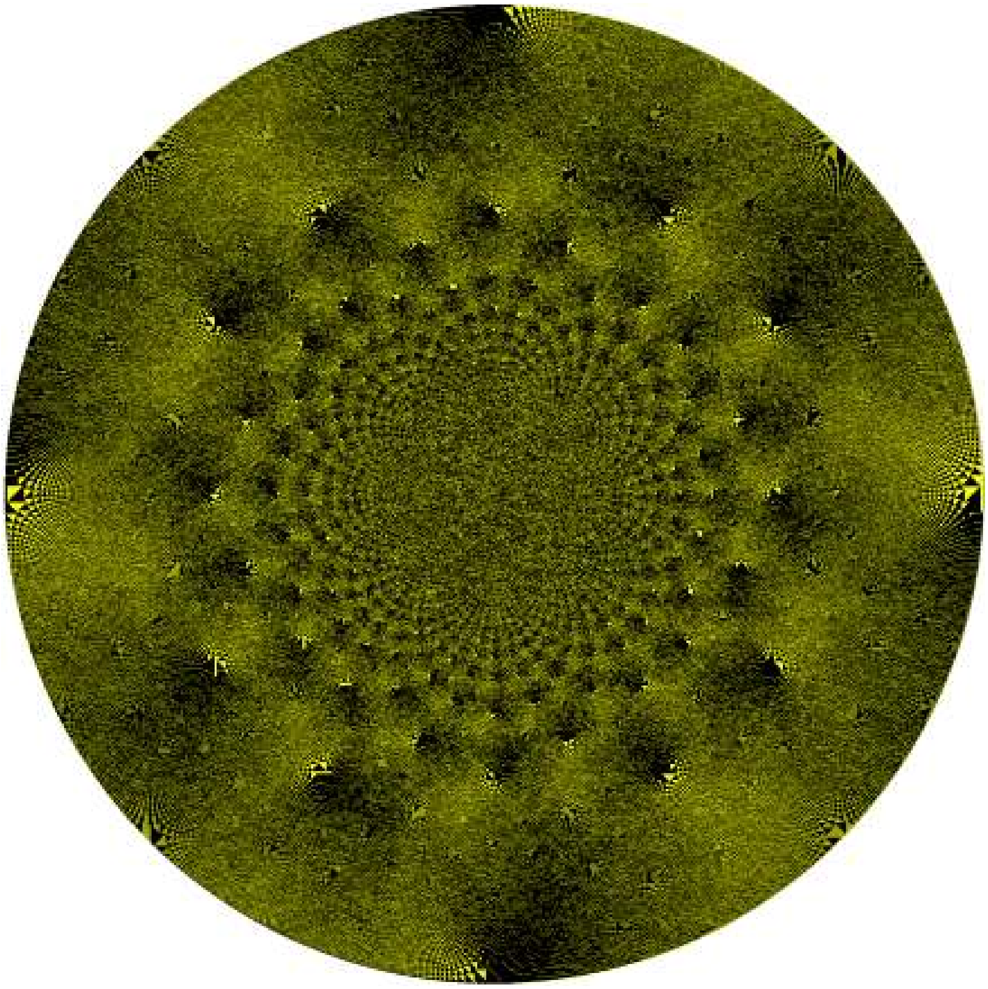


Figure A.2.: The set $A_{3,000,000}$ for quasirandom IDLA on $\mathbb{Z}^2$, again, with initial rotor state $\rightarrow$ for all fields and rotor sequence $(\rightarrow, \downarrow, \leftarrow, \uparrow)$. Only fields with $\downarrow$ rotor state are colored yellow and all others are colored black to enhance the effect of patterns that can be found in the aggregation model.

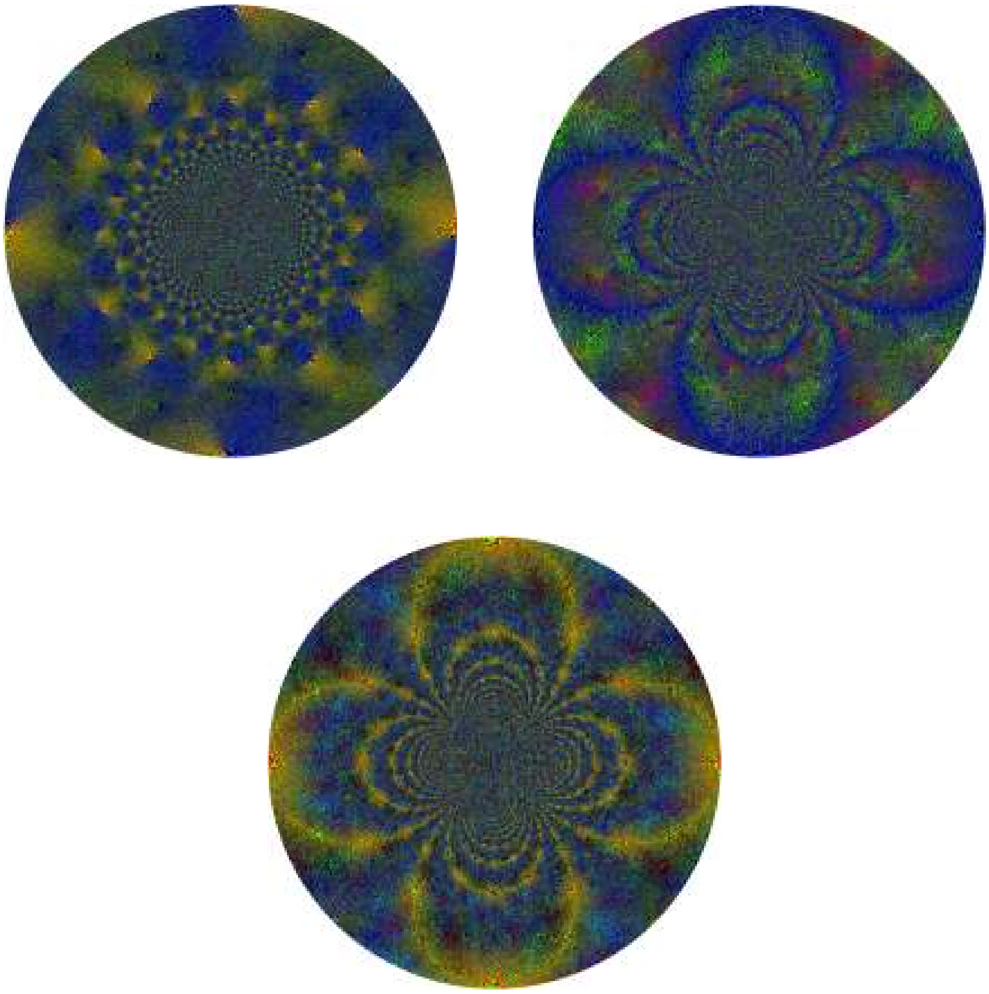
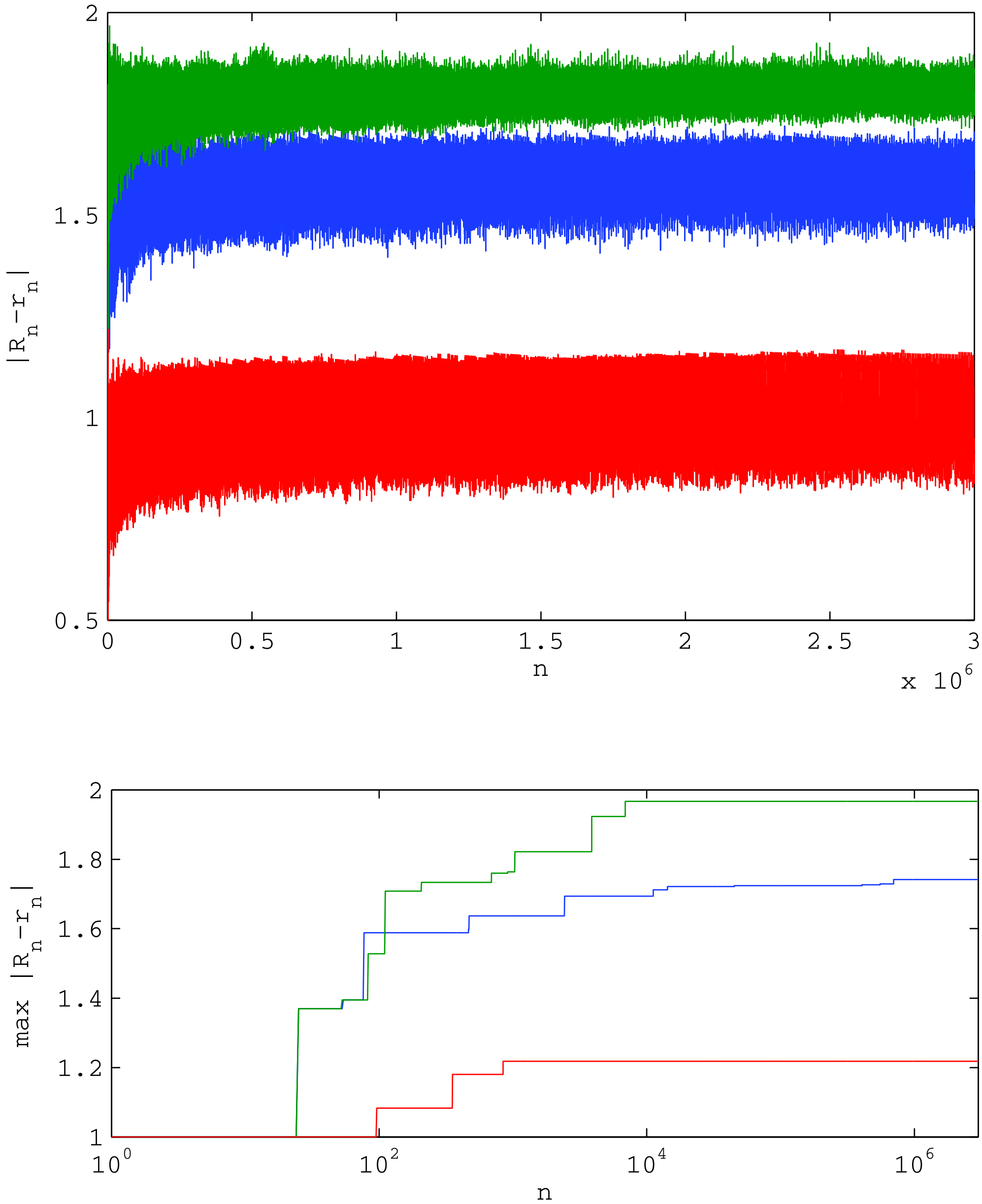


Figure A.3.: The set $A_{3,000,000}$ for quasirandom IDLA on $\mathbb{Z}^2$ comparing the three principal rotor sequences $(\rightarrow, \downarrow, \leftarrow, \uparrow)$, $(\rightarrow, \downarrow, \uparrow, \leftarrow)$, and $(\rightarrow, \leftarrow, \downarrow, \uparrow)$. All fields have initial rotor state $\rightarrow$ and are colored according to their rotor direction as in Figure A.1.

Top left: $(\rightarrow, \downarrow, \leftarrow, \uparrow)$, top right: $(\rightarrow, \downarrow, \uparrow, \leftarrow)$, bottom: $(\rightarrow, \leftarrow, \downarrow, \uparrow)$.



— (→, ↓, ←, ↑), — (→, ↓, ↑, ←), and — (→, ←, ↓, ↑)

Figure A.4.: The function $|R_n - r_n|$ for quasirandom IDLA on $\mathbb{Z}^2$ for the three principal rotor sequences (→, ↓, ←, ↑), (→, ↓, ↑, ←), and (→, ←, ↓, ↑). The bottom graph gives the maximum radius difference up to n in logarithmic scale for n.

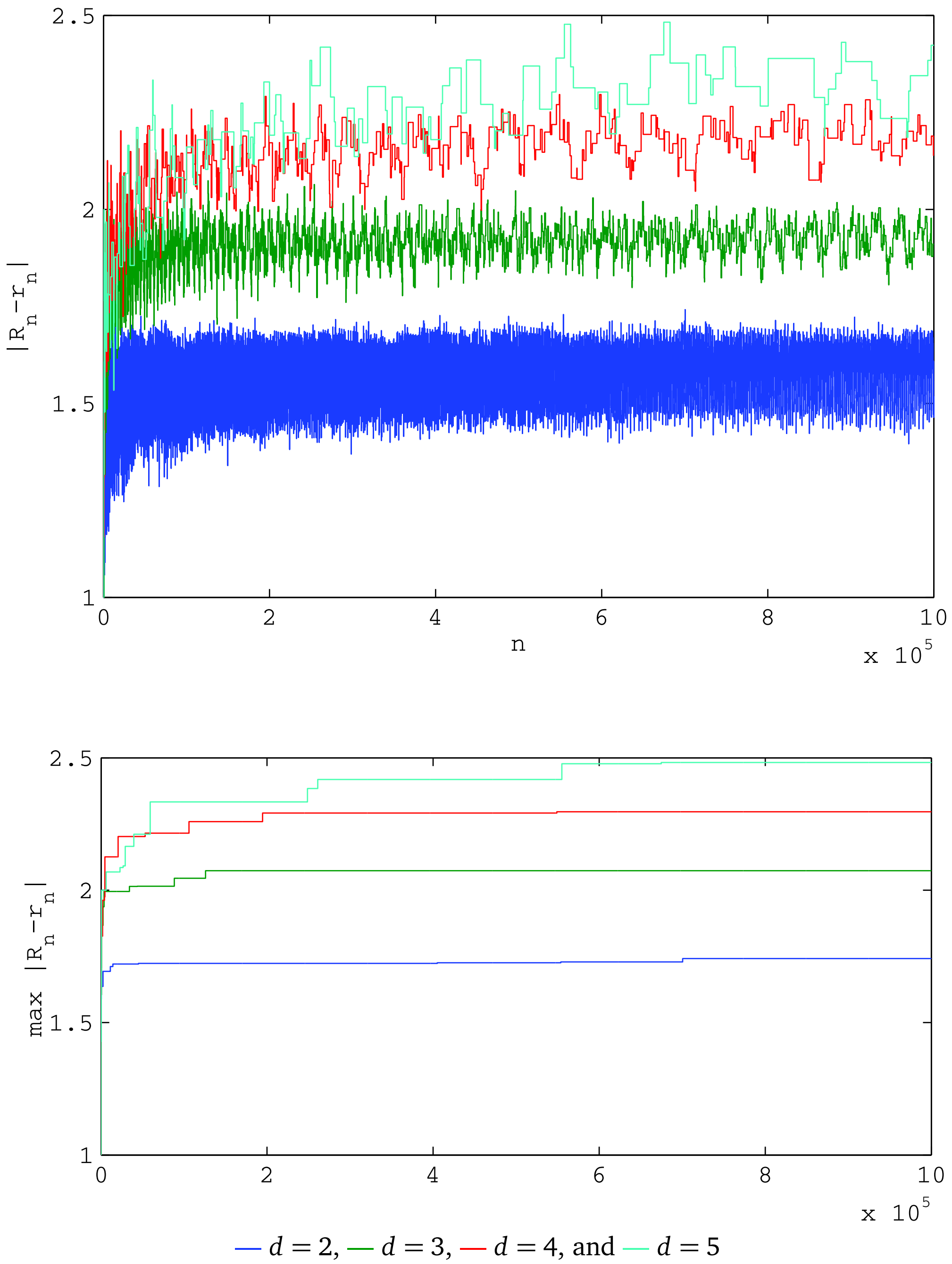


Figure A.5.: The function $|R_n - r_n|$ for quasirandom IDLA on $\mathbb{Z}^d$ for $n = 1, \ldots, 1000000$ and $d \in \{2, 3, 4, 5\}$. All models use rotor sequences of the form

$$((1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1),(-1,0,\ldots,0),\ldots,(0,0,\ldots,-1))$$

in vector notation, where the first entry is also the initial rotor direction for all fields. The bottom graph gives the maximum radius difference up to n.

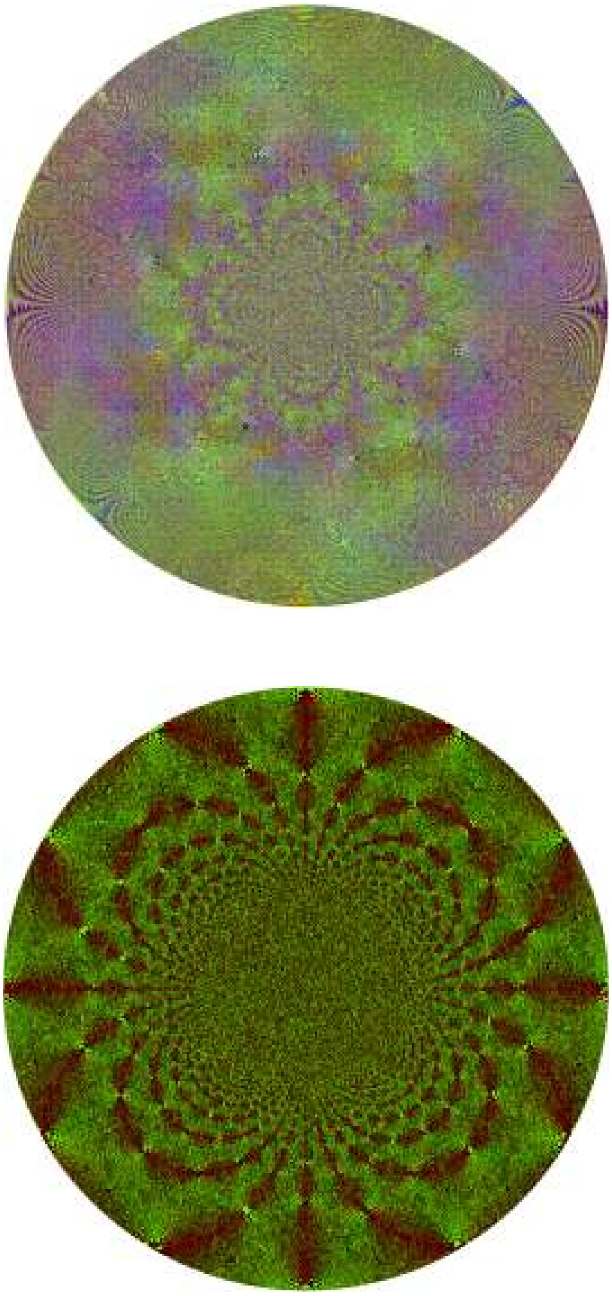


Figure A.6.: The set $A_{3,000,000}$ for quasirandom IDLA with different underlying grids as defined in Section 4.3, all using cyclical rotor sequences and one single initial rotor state for all fields. The colors indicate rotor direction.

Top: diagonal grid, bottom: triangle grid.

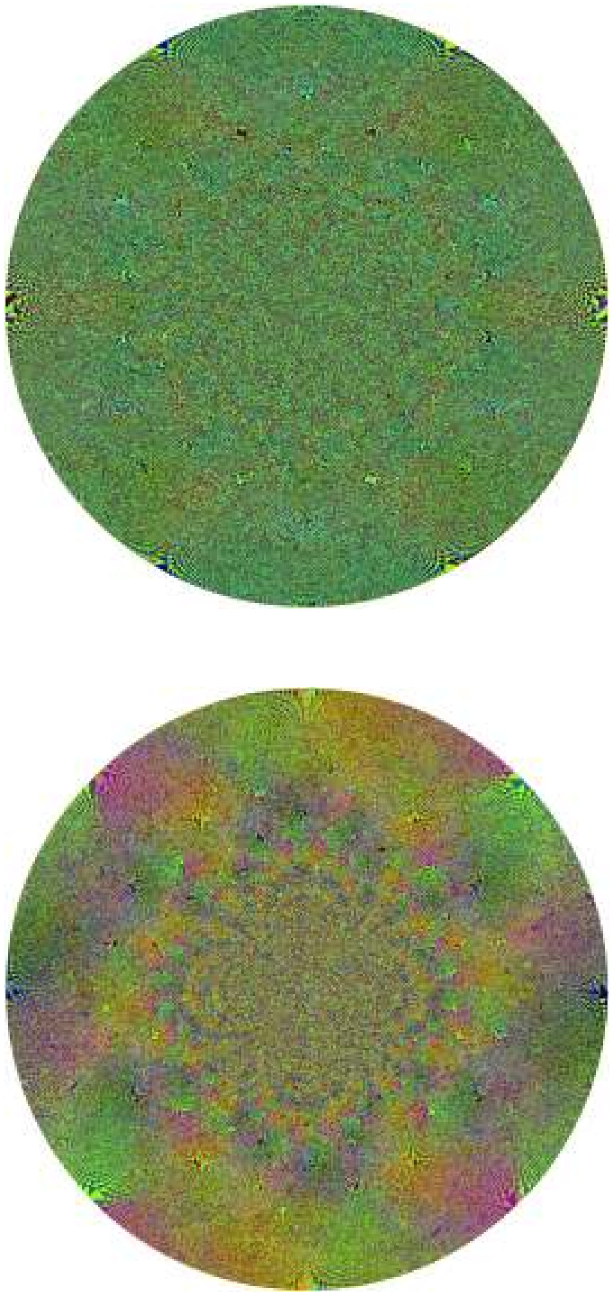


Figure A.7.: Data shown is as in Figure A.6.
Top: hexagon grid, bottom: octagon-square grid.

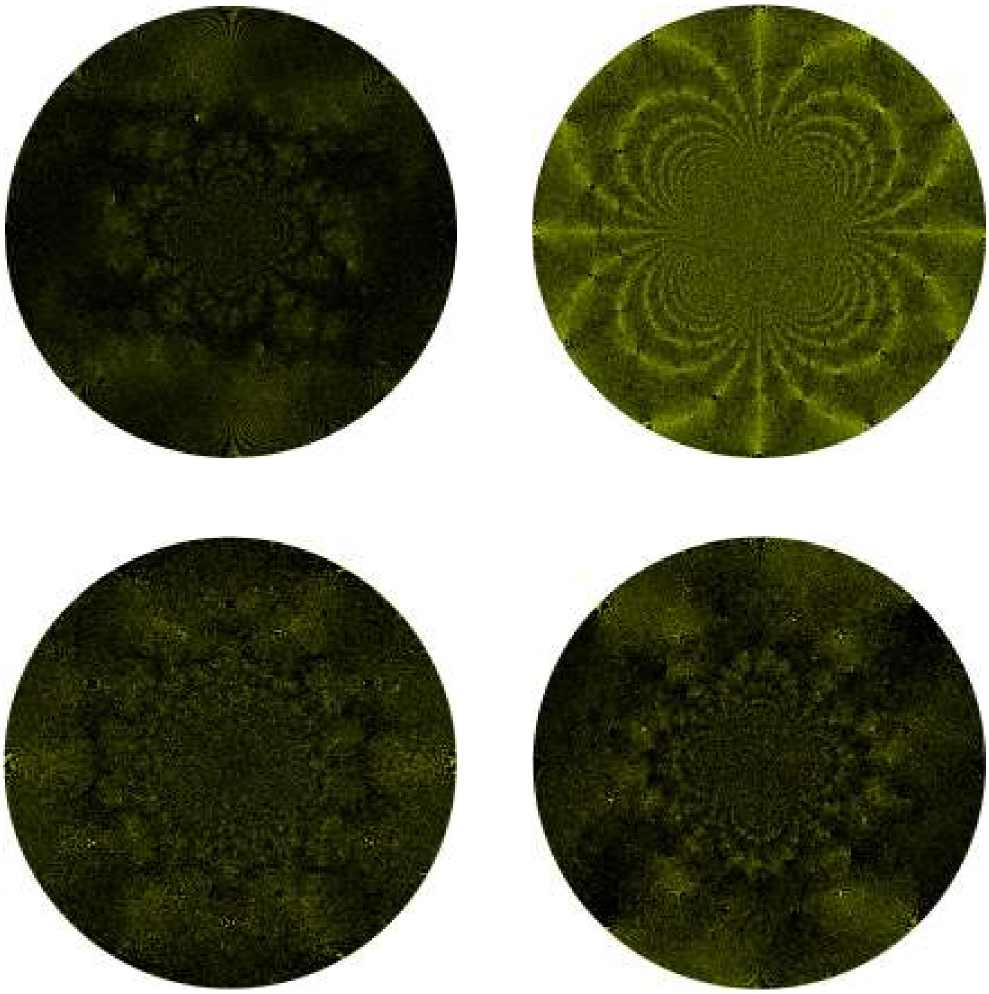
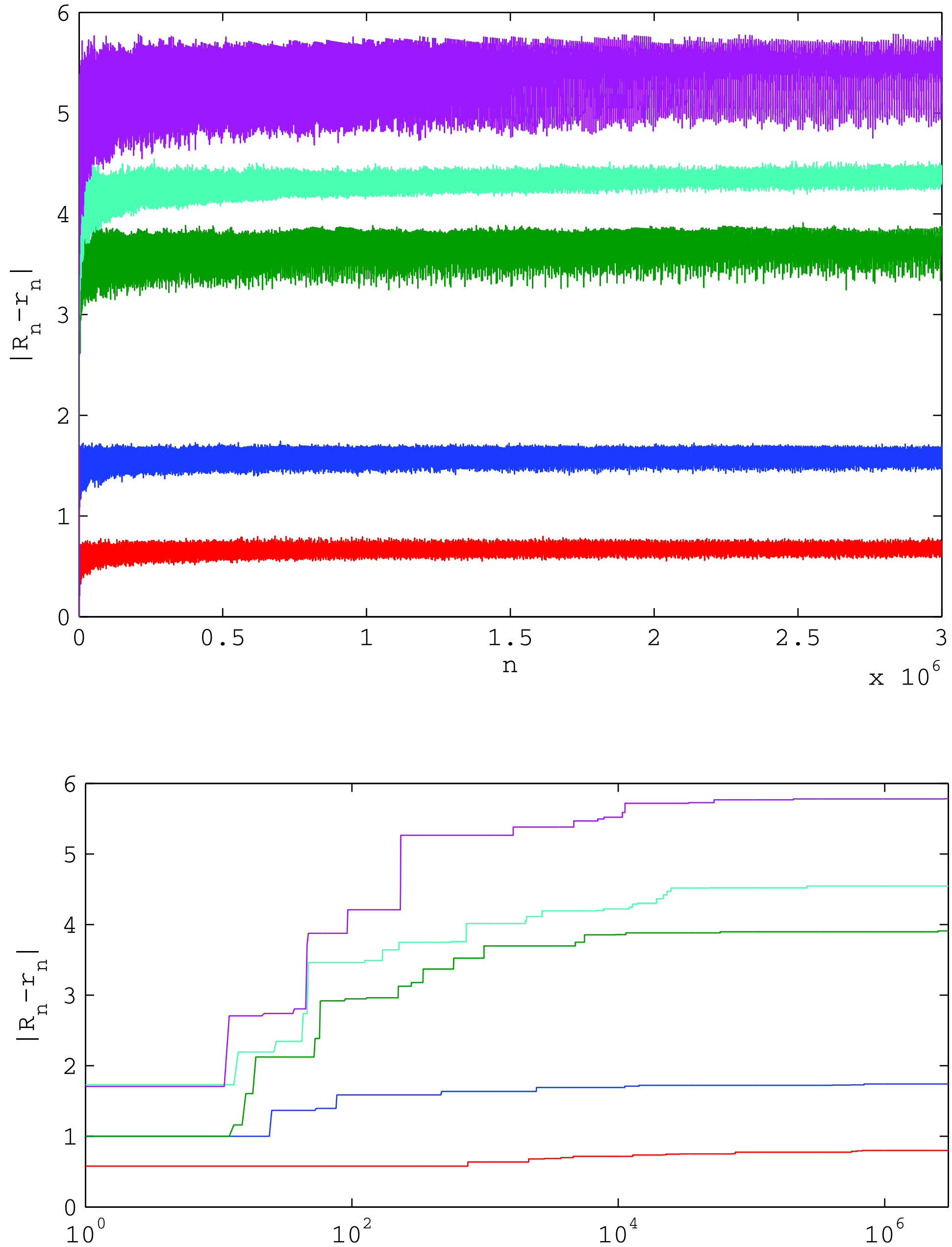


Figure A.8.: Data shown is as in Figure A.6, only rotors pointing in one direction are colored yellow, all others black.

Top left: diagonal grid, top right: triangle grid, bottom left: hexagon grid, bottom right: octagon-square grid.



— Classical, — diagonal, — triangle, — hexagon, and — octagon-square grid

Figure A.9.: The function $|R_n - r_n|$ for quasirandom IDLA with different underlying grids for $n = 1, \dots, 3000000$. All models use cyclical rotor sequences and one single initial rotor state for all fields. The bottom graph gives the maximum radius difference up to n in logarithmic scale for n.

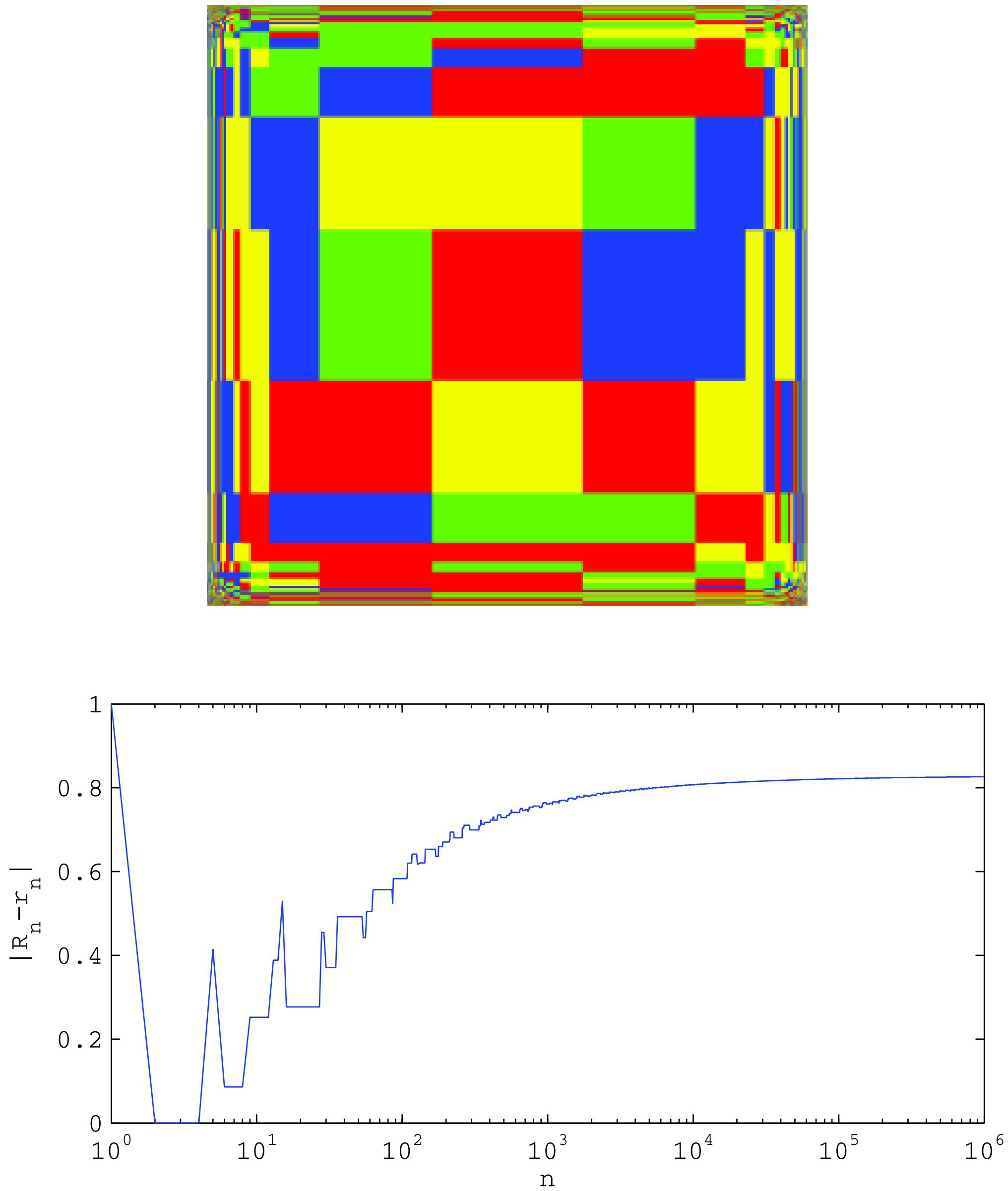


Figure A.10.: The set $A_{1,000,000}$ for quasirandom IDLA on the harmonic grid as defined in Section 4.3, using one single initial rotor state for all fields. The colors indicate rotor direction. The bottom graphs shows $|R_n - r_n|$ for $n = 1, \ldots, 1000000$ with logarithmic scale for n.

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