Homogeneous Structures and Ramsey Classes

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In this project report we discuss the relation between Ramsey classes and homogeneous structures and give various examples for both of them. In addition, we explain the role of orderings for Ramsey classes and study the Ramsey properties of permutations.

1 Introduction

The finite Ramsey theorem states that for all natural numbers a, b and r there is a natural number c such that in any r-coloring of the a-element subsets of the set $\{1, \dots, c\}$ there is a monochromatic b-element subset. In the following the number c will be denoted by $R(a, b, r)$.

Different variants and generalizations of this result (cf. Nešetřil, 1995) have been investigated. The concept of Ramsey classes is the most general of these. Roughly speaking, a Ramsey class is a class $\mathcal{K}$ of objects such that for each natural number r and each choice of objects $A, B \in \mathcal{K}$ there is an object $C \in \mathcal{K}$ such that an analog of the main part of Ramsey's theorem holds. Here, A plays the role of a, B the role of b, and C the role of c.

In this report, we define what Ramsey classes are, give various examples of them and explain their relation to homogeneous structures and orderings. In particular we will discuss the Ramsey properties of permutations.

The remainder of this report is structured as follows. In Section 2 we will formally define Ramsey classes and in Section 3 we will introduce homogeneous structures. In Section 4 we will discuss orderings and in Section 5 the connection between Ramsey classes and homogeneous structures. In Section 6 finally, we will consider permutations.

2 Ramsey Classes

The definition of a Ramsey class can be stated for general classes of objects endowed with subobjects and isomorphisms. So, consider a class $\mathcal{K}$ of objects and assume that subobjects of objects in $\mathcal{K}$ and isomorphisms between objects from $\mathcal{K}$ are well defined. Moreover, let $\mathcal{K}$ be closed under isomorphisms. Let $A, B, B' \in \mathcal{K}$. If A is a subobject of B we write $A \to B$ and if B and B' are isomorphic we write $B \cong B'$. Moreover, $\binom{B}{A}$ is the set of all subobjects A' with $A' \to B$ and $A' \cong A$.

Definition: *For $A, B, C \in \mathcal{K}$ and $r \in \mathbb{N}$ the* Erdős-Rado partition arrow $C \to (B)^A_r$ *denotes the following: For each partition $\binom{C}{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_r$ there exist $i \in \{1, \ldots, r\}$ and $B' \to C$ such that $B' \cong B$ and $\binom{B'}{A} \subset \mathcal{A}_i$. We also call B' a* monochromatic *copy of B in C and the partition of $\binom{C}{A}$ a* coloring.

For $A \in \mathcal{K}$ say that $\mathcal{K}$ is A-Ramsey *if for all $B \in \mathcal{K}$ and $r \in \mathbb{N}$ there exists a $C \in \mathcal{K}$ such that $C \to (B)^A_r$.*

The class $\mathcal{K}$ is a Ramsey class *if it is A-Ramsey for all $A \in \mathcal{K}$.*

It remains to define which objects we consider as subobject of other objects. One natural aproach is to use embeddings. In this report we consider only classes of relational structures. For $I \subset \mathbb{R}$ let $\Delta = (\delta_i)_{i \in I}$ be a sequence of natural numbers. Then $Rel(\Delta)$ denotes the set of all *relational structures* $A = (X, (R_i)_{i \in I})$ where X is a set and $R_i \subset X^{\delta_i}$. We also call Δ the *type* or *signature* of $(X, (R_i)_{i \in I})$ and write $\underline{A}$ for X. Let $A = (X, (R_i)_{i \in I})$ and $A' = (X', (R'_i)_{i \in I'})$ be two relational structures of types $\Delta = (\delta_i)_{i \in I}$ and $\Delta' = (\delta'_i)_{i \in I'}$, respectively. An injective mapping $f : X \to X'$ is an embedding of A into A' if $I \subset I'$ and $\delta_i = \delta'_i$ for $i \in I$ and $(x_j)_{j=1,\ldots,\delta_i} \in R_i$ iff $(f(x_j)_{j=1,\ldots,\delta'_i}) \in R'_i$. Accordingly, A is a *subobject* or *substructure* of A' if there is an inclusion embedding f of A into A', i.e. if $X \subset X'$ and f is the identity on X. A bijective embedding is called *isomorphism*.

2.1 Examples of Ramsey Classes

The finite Ramsey theorem asserts that finite sets form a Ramsey class. Here, we name some more examples. In the following chapters, some of them will be discussed in greater detail.

The following classes of finite structures are Ramsey classes:
(cf. Nešetřil, 1995; Nešetřil, 2004; Nešetřil, 2005)

1. all ordered metric spaces,
2. all (lineraly ordered) relational structures of a fixed type,
3. all (lineraly ordered) graphs,
4. all partially ordered sets (together with a fixed linear extension),

5. all finite vector spaces over a fixed field,
6. all (lineraly ordered) partitions.

Here, a *partition* is a complete k-partite graph for some k. We say that a structure A is *linearly ordered* if it is endowed with an additional linear ordering of $\underline{A}$ (cf. Section 4).

3 Homogeneous Structures

As was observed by Nešetřil, 1989, there is a strong connection between Ramsey classes and homogeneous structures. Before turning to these results in Section 5 we first need to review some model theoretic concepts.

Definition: *A relational structure B is called* homogeneous *if every isomorphism $f : A \to A'$ between finite substructures A and A' of B can be extended to an automorphism of B.*

For an infinite structure B the age *of B (denoted by $age(B)$) is the class of all structures embeddable in B. We also say that B is* universal *for $age(B)$.*

Homogeneity is a very strong restriction to relational structures and enforces high symmetry. Accordingly there are not many finite examples. Studying infinite homogeneous structures, however, turned out to be more interesting. A classical result by Fraïssé gives a characterization of homogeneous structures in terms of their age. For this we need some more definitions.

Definition: *Let $\mathcal{K}$ be a class of relational structures. Then $\mathcal{K}$ is* hereditary *if $A' \in \mathcal{K}$ and $A \to A'$ imply $A \in \mathcal{K}$.*

The class $\mathcal{K}$ has the joint embedding property *if for any $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that both A and B are embeddable in C. We say, that $\mathcal{K}$ has the* amalgamation property *if for every $A, B, B' \in \mathcal{K}$ and every embedding $f : A \to B$ of A into B and every embedding $f' : A \to B'$ of A into B' there exists $C \in \mathcal{K}$ and embeddings $g : B \to C$ and $g' : B' \to C$ such that $g \circ f = g' \circ f'$ (see Figure 3). We call (C, g, g') an* amalgamation *of the* amalgam *(A, B, B', f, f'). If the intersection of $g(\underline{B})$ and $g'(\underline{B'})$ is equal to $g \circ f(\underline{A})$ we say that the amalgamation is* strong. Strong joint embeddings *are defined analogously.*

We can now state the result of Fraïssé which gives a necessary and sufficient condition for a class of finite structures to be the age of a countable homogeneous relational structure.

Theorem 3.1 (Fraïssé, 1953)
A class $\mathcal{K}$ of finite relational structures is the age of a countable homogeneous relational structure U iff the following four conditions hold:

1. *$\mathcal{K}$ has countably many isomorphism classes,*
2. *$\mathcal{K}$ is closed under isomorphism,*

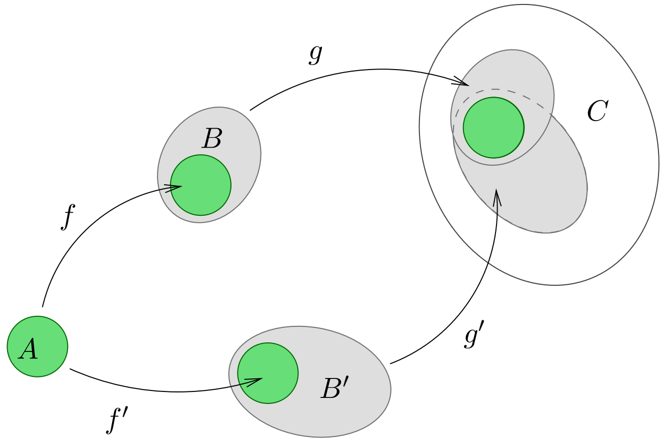

Figure 1: (C, g, g') is an amalgamation of the amalgam (A, B, B', f, f').

3. *$\mathcal{K}$ is closed under substructures,*

4. *$\mathcal{K}$ has the amalgamation property.*

The structure U is also called *generic* for $\mathcal{K}$ or the *Fraïssé limit* of $\mathcal{K}$. Observe that the first three conditions are trivial since they are satisfied for the age of any countable structure. So the essential property of ages of homogeneous structures is the amalgamation property.

3.1 Examples of Homogeneous Structures

The homogeneous structures have been classified for a number of important classes of relational structures.

We start with graphs. For a class of graphs $\mathcal{K}$ $\overline{\mathcal{K}}$ denotes the class of all complements of graphs in $\mathcal{K}$. Moreover, let $\mathcal{COMPL} := \{K_i : i \in \mathbb{N}\}$ be the class of all finite complete graphs, $\mathcal{EQ}$ be the class of all finite equivalences, i.e., disjoint unions of complete graphs, and K_k-$FREE$ the class of all finite K_k-free graphs. Let $\mathcal{EQ}_k$ be the class of all finite euqivalences where each equivalence class has size at most k and $\mathcal{EQ}^k$ be the class of all finite equivalences with at most k equivalence classes.

***Theorem 3.2* (Lachlan & Woodrow, 1980)**
Each countable infinite homogeneous undirected graph is the Fraïssé limit of one of the following classes of finite graphs:

1. *$\mathcal{COMPL}$ (all complete graphs),*

2. *$\mathcal{EQ}$ (all equivalences),*

3. *$\mathcal{EQ}_k$ or $\mathcal{EQ}^k$ for each k (all bounded equivalences),*

4. K_k-*FREE for each* k *(all* K_k *free graphs),*

5. $\mathcal{GRAPH}$ *(all graphs),*

6. $\overline{\mathcal{K}}$ *for each of the above classes* $\mathcal{K}$*.*

It follows that each other hereditary graph class, e.g. planar graphs or graphs of girth at least five, does not have the amalgamation property.

The *generic graph*, i.e., the Fräissé limit of the class $\mathcal{GRAPH}$, is the *random graph*, also called the *Rado graph*.

The directed countable homogeneous graphs have been classified by Cherlin, 1998. There are uncountably many of them. Here, we concentrate on some special cases: tournaments and partially ordered sets.

Lachlan, 1984 characterized the countable homogeneous tournaments. Let $C_3^{\circlearrowleft}$ be an oriented K_3 and let $\mathcal{TOUR}$ be the class of all finite tournaments and by $\mathcal{TRANS}$ the class of all transitive tournaments. A Tournament $T = (V, R)$ is a *local order* if for all vertices $v \in V$ both, the set $\{w : vw \in R\}$ and the set $\{w : wv \in R\}$ induce a transitive tournament in T. $\mathcal{LORD}$ will denote the set of all local orders.

Theorem 3.3 (Lachlan, 1984)
Each countable infinite homogeneous tournament is the Fraïssé limit of one of the following classes of finite tournaments:

1. $\{C_3^{\circlearrowleft}\}$ *(the oriented triangle),*

2. $\mathcal{TRANS}$ *(all transitive tournaments),*

3. $\mathcal{LORD}$ *(all local orders),*

4. $\mathcal{TOUR}$ *(all tournaments).*

The classification programme for partially ordered sets has already been completed by Schmerl, 1979. $\mathcal{ORD}$ denotes the class of all finite linear orders, $\mathcal{CH}$ the class of all finite disjoint unions of finite linear orders, and $\mathcal{CH}_k$ those partial orders from $\mathcal{CH}$ that are unions of at most k linear orders. A *chain-sum* between two disjoint antichains $A_1 = (V_1, \emptyset)$ and $A_2 = (V_2, \emptyset)$ is the partial order $P = (V_1 \cup V_2, R)$ where $R := \{v_1 v_2 : v_1 \in V_1, v_2 \in V_2\}$. This definition generalizes to chainsums of k antichains. Then, Σ is the class of all finite chain-sums between finite antichains and Σ_k contains those members of Σ that are chain-sums of at most k antichains.

Theorem 3.4 (Schmerl, 1979)
Each countable infinite homogeneous partially ordered set is the Fraïssé limit of one of the following classes of finite partially ordered sets:

1. $\mathcal{ORD}$ *(all linear orders),*

2. $\mathcal{CH}_k$ *for each* k *(all unions of at most* k *chains),*

3. *$\mathcal{CH}$ (all unions of chains),*

4. *$\mathcal{ANTI}$ (all antichains),*

5. *Σ_k for each k (all chain-sums of at most k antichains),*

6. *Σ (all chain-sums of antichains),*

7. *$\mathcal{POSET}$ (all partially ordered sets).*

4 Orderings

Which properties do Ramsey classes have in common? In the next section we will see that all Ramsey classes have the amalgamation property. Before explaining this in greater detail we will investigate the role of rigidity and symmetry in this section. In particular we will show that orderings are important for studying Ramsey properties.

According to the definition given in Section 2, substructures correspond to inclusion embeddings. However, with this definition many interesting classes of structures are A-Ramsey only for highly symmetric objects A. We will illustrate this for graphs.

Consider the class of all graphs with induced subgraphs as substructures. Let P_3 be the path of length 3 and C_5 be the cycle of length 5. Then there is no graph G such that $G \to (C_5)_k^{P_3}$ and consequently the class of all graphs is not P_3-Ramsey. Indeed, let G be a graph and $\leq_G$ be an arbitrary ordering of its vertices. Now, choose the following coloring of $\binom{G}{P_3}$: If $P_3' \in \binom{G}{P_3}$ is monotone with respect to $\leq_G$ then color P_3' with color 1, otherwise with color 2. But in each linear ordering of C_5 we have both, a monotone copy of P_3 and a non-monotone copy of P_3. Therefore there is no monochromatic copy of C_5 in G. The next lemma shows that this construction is also possible for other graphs than P_3 and C_5 (cf. Nešetřil, 1995).

Lemma 4.1 (Ordering Lemma)
Let F be an undirected graph and $\leq_F$ a linear orderings of the vertices of F. Then there is a graph H such that for every ordering $\leq_H$ of the vertices of H there is an embedding $f : F \to H$ that is monotone with respect to $\leq_F$ and $\leq_H$, i.e., $u \leq_F v$ iff $f(u) \leq_H f(v)$ for $u, v \in \underline{F}$.

In this lemma we considered the class $\mathcal{K}$ of undirected graphs G together with the class $\preceq$ of all linear orderings $\leq_G$ of their vertices. A similar result can be shown for various other classes $\mathcal{K}$ of structures together with classes $\preceq$ of corresponding orderings, e.g., partially ordered sets together with their linear extensions or metric spaces together with all possible orderings of their points (cf. Nešetřil, 2004). In this case we also say that $\mathcal{K}$ has the *ordering property* with respect to $\preceq$.

Observation 4.2
Let $\mathcal{K}$ be a class of relational structures that has the strong joint embedding property and the ordering property with respect to the class $\preceq$ of all orderings of its structures.

If $A \in \mathcal{K}$ and there is an injective mapping $f : \underline{A} \to \underline{A}$ that is not an automorphism of A then $\mathcal{K}$ is not A-Ramsey.

This can be shown by choosing a structure B that contains two disjoint copies of A_1 and A_2. B exists by the strong joint embedding property. Let $\leq_A^1$ be an arbitrary ordering of $\underline{A}$ and $\leq_A^2$ be such that $x \leq_A^1 y$ iff $f(x) \leq_A^2 f(y)$ for $x, y \in \underline{A}$. Now, let $\leq_B$ be an ordering of $\underline{B}$ that is monotone with respect to $\leq_A^1$ for A_1 and to $\leq_A^2$ for A_2. Then we can proceed in analogy to the discussion for P_3 and C_5 above. It follows that the class of undirected graphs is not F-Ramsey for all graphs F that are neither complete nor empty.

The following more general result was proven in Nešetřil, 2005. We say that a structure $A \in \mathcal{K}$ is *order invariant* with respect to a class $\preceq$ of orderings if for any choice of orderings $\leq_A^1, \leq_A^2 \in \preceq$ of $\underline{A}$ there is an isomorphism $f : \underline{A} \to \underline{A}$ that is monotone with respect to $\leq_A^1$ and $\leq_A^2$.

Theorem 4.3 (Nešetřil, 2005)
If $\mathcal{K}$ has the ordering property with respect to $\preceq$ and if $\mathcal{K}$ is A-Ramsey for $A \in \mathcal{K}$, then A is order invariant with respect to $\preceq$.

Another possibility is to consider substructures as embeddings themselves: Now, each embedding $A \to B$ of a structure A into a structure B is a substructure of B. (Consider two embeddings $A \to B$ and $A' \to B$ as equivalent if $A = A'$. Then the equivalence classes of the resulting equivalence relation correspond to the substructures of the original definition.) With this definition, however, non-rigid structures, i.e., structures that have a non-trivial automorphism, give rise to non-Ramsey classes.

Observation 4.4
Let $\mathcal{K}$ be a class of relational structures and $A \in \mathcal{K}$ have a non-trivial automorphism. Then $\mathcal{K}$ does not have the A-Ramsey property.

For this, observe that for each copy A' of A in a structure $C \in \mathcal{K}$ there are two distinct embeddings of A in $\underline{A'}$. So, just assign different colors to these two embeddings.

Accordingly with this definition of substructures many classes of relational structures are trivially not Ramsey. We therefore need to find another solution.

Observations 4.2 and 4.4 motivate to extend relational structures by orderings of their vertices and to consider only monotone embeddings: Let $\mathcal{K}$ be a class of relational structures and $\preceq$ be a class of orderings of the structures in $\mathcal{K}$. Then , $(\mathcal{K}, \preceq)$ contains all objects of the form $(A, \leq_A)$ where $A \in \mathcal{K}$ and $\leq_A \in \preceq$ is an ordering of $\underline{A}$. In addition, $(A, \leq_A)$ is a subobject of $(B, \leq_B)$ if there is an embedding $f : A \to B$ that is monotone with respect to $\leq_A$ and $\leq_B$. It follows that structures in $(\mathcal{K}, \preceq)$ are rigid. Moreover, since each object $(A, \leq_A)$ in $(\mathcal{K}, \preceq)$ is endowed with a unique ordering $\leq_A$, each $(A, \leq_A)$ is trivially order invariant (with respect to these orderings). Accordingly we avoid both problems discussed above.

Structures	Ramsey Classes
graphs	$\mathcal{COMPL}$, $(\mathcal{EQ}, \preceq)$, $(K_k\text{-}FREE, \preceq)$, $(\mathcal{GRAPH}, \preceq)$ [1] (Nešetřil, 1989)
tournaments	$\mathcal{TRANS}$, $(\mathcal{TOUR}, \preceq)$ (cf. Nešetřil, 2005)
partially ordered sets	$\mathcal{ORD}$, $(\mathcal{CH}, \preceq)$, $\mathcal{ANTI}$, $(\Sigma, \preceq)$, $(\mathcal{POSET}, \preceq)$ (cf. Nešetřil & Rödl, 1984; Nešetřil, 2005)

Table 1: The Ramsey classes for graphs, tournaments and partially ordered sets.

Observe, for example, that the class $\mathcal{GRAPH}$ of all finite graphs is neither Ramsey for our original definition of substructures as embeddings nor for the definition that groups embeddings into equivalence classes. The class

$$(\mathcal{GRAPH}, \preceq) := \{(G, \leq_G) : G \in \mathcal{GRAPH}, \leq_G \text{ is a linear ordering of } \underline{G}\},$$

however, is a Ramsey class as we will see in the next section.

5 The Connection between Ramsey Classes and Homogeneous Structures

In the last section, we saw that the structures in Ramsey classes are rigid. But this alone is not sufficient. Another necessary condition is amalgamation.

Theorem 5.1 (cf. Nešetřil, 2004)
If $\mathcal{K}$ is a Ramsey class then $\mathcal{K}$ has the amalgamation property.

By Fraïssé's Theorem (Theorem 3.1) we can therefore concentrate on homogeneous structures when searching for Ramsey classes.

Corollary 5.2 (Nešetřil, 2004)
If $\mathcal{K}$ is a hereditary isomorphism-closed Ramsey class then $\mathcal{K}$ is the age of a homogeneous structure.

Moreover, there is a stronger version of Theorem 5.1 that involves orderings. A class $\preceq$ of orderings of structures from $\mathcal{K}$ is called *rich* if the following holds: Let $A, B, B' \in \mathcal{K}$ and $f : A \to B$ be an embedding of A into B and $f' : A \to B'$ an embedding of A into B'. Then there are $\tilde{B}, \tilde{B}' \in \mathcal{K}$, extensions $\tilde{f}$ and $\tilde{f}'$ of f and f', respectively, and orderings $\leq_A, \leq_{\tilde{B}}, \leq_{\tilde{B}'} \in \preceq$ of A, $\tilde{B}$, and $\tilde{B}'$, respectively, such that every amalgamation of $((A, \leq_A), (\tilde{B}, \leq_{\tilde{B}}), (\tilde{B}', \leq_{\tilde{B}'}), \tilde{f}, \tilde{f}')$ in $(\mathcal{K}, \preceq)$ is strong.

[1] and in addition the classes $\overline{\mathcal{K}}$ and $(\overline{\mathcal{K}}, \preceq)$ for all the listed classes $\mathcal{K}$ and $(\mathcal{K}, \preceq)$, respectively

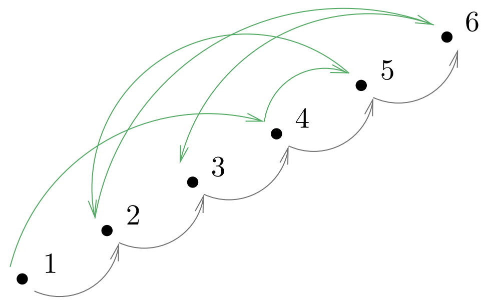

Figure 2: A representation of the permutation with pattern $1, 4, 5, 2, 6, 3$ by two linear orders (transitive arcs are omitted).

***Theorem 5.3** (**Nešetřil, 2004**)*
If $\preceq$ is a rich class of orderings for $\mathcal{K}$ and $(\mathcal{K}, \preceq)$ is a Ramsey class then $\mathcal{K}$ has the strong amalgamation property.

In Table 1 we indicate which of the homogeneous structures introduced in Section 3.1 give rise to a Ramsey class. By the remarks in Section 4 it is necessary to endow some of these classes $\mathcal{K}$ with orderings $\preceq$. For graphs and tournaments $\preceq$ contains all possible orderings of objects $\mathcal{K}$, for posets all linear extensions of objects from $\mathcal{K}$.

6 Permutations

There are different ways of defining permutations. In this report we use the definition introduced by Cameron, 2002. One advantage of this definition is that it does not only apply to finite permutations but also to infinite ones.

Definition: *A* permutation $P = (X, <_1, <_2)$ *consists of a set* $X \subset \mathbb{R}$ *and two linear orders* $<_1$ *and* $<_2$ *of* X *where* $<_1$ *is the natural order on* X *and* $<_2$ *is arbitrary. An example is shown in Figure 6. A* subpermutation *of* P *is a permutation* $P' = (X', <'_1, <'_2)$ *with* $X' \subseteq X$ *such that* $<'_2$ *is the restriction of* $<_2$ *to* X'*. Let* $P = (X, <_1, <_2)$ *be a finite permutation with* $|X| = x$ *and consider a sequence* $(p_i)_{i=\{1,...,x\}}$ *of pairwise distinct numbers* $p_i \in \{1, \ldots, x\}$*. Then we call* (p_i) *the* pattern *of* P *if the permutation* $P' = (\{1, \ldots, x\}, <'_1, <'_2)$ *with* $i < j$ *iff* $p_i <'_2 p_j$ *for* $i, j \in \{1, \ldots, x\}$ *is isomorphic to* P*.*

The countable homogeneous permutations were characterized by Cameron, 2002. An *identity permutation* is a permutation of the form $(X, <_1, <_1)$ and a *reversal* is a permutation $(X, <_1, <_2)$ where $<_2$ is the reversed natural order. An *increasing sequence of decreasing sequences* is a permutation that contains no subpermutation with pattern $2, 3, 1$ and no subpermutation with pattern $3, 1, 2$. *Decreasing sequences of increasing sequences* are defined analogously (no $2, 1, 3$ and no $1, 3, 2$).

Theorem 6.1 (Cameron, 2002)
The age of each countable infinite homogeneous permutation is one of the following classes of finite permutations:

1. *the class of identity permutations,*
2. *the class of reversals,*
3. *increasing sequences of decreasing sequences,*
4. *decreasing sequences of increasing sequences ,*
5. *the class of all permutations.*

As a direct consequence of the finite Ramsey theorem we have the following.

Observation 6.2
The class of identity permutations and the class of reversals are Ramsey classes.

In the next two sections we will study the remaining three classes from theorem 6.1.

6.1 Increasing and Decreasing Sequences and the Product Ramsey Theorem

First, we consider the class of increasing sequences of decreasing sequences and the class of decreasing sequences of increasing sequences. For showing that they are Ramsey we make use of the *product Ramsey theorem*.

Theorem 6.3 (Product Ramsey Theorem)
For all $r, p, t, n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that the following holds: If

- *$X_1, X_2, \ldots X_t$ are pairwise disjoint sets,*
- *$|X_i| \geq N$ for all $1 \leq i \leq t$,*
- *we r-color $\binom{X_1}{p} \times \cdots \times \binom{X_t}{p}$,*

then there are $Y_i \subset X_i$ with $|Y_i| \geq n$ for all $1 \leq i \leq t$ such that $\binom{Y_1}{p} \times \cdots \times \binom{Y_t}{p}$ is monochromatic.

Let $PR(r, p, t, n)$ denote the number N given by this theorem for input r, p, t, n. We need a stronger version of this result.

For this, we consider sets $A \subset \mathbb{N}^2$ of pairs. Two finite sets $A \subset \mathbb{N} \times J$ and $A' \subset \mathbb{N} \times J'$ with $J = \{j_1, \ldots, j_n\} \subset \mathbb{N}$ and $J'\{j'_1, \ldots, j'_{n'}\} \subset \mathbb{N}$ where $j_1 < \ldots < j_n$ and $j'_1 < \ldots < j'_{n'}$ are isomorphic if $n = n'$ and $|A \cap (\mathbb{N} \times \{j_i\})| = |A' \cap (\mathbb{N} \times \{j'_i\})|$ for all $1 \leq i \leq n$. Moreover, A' is a substructure of B if it is a subset of B and $\binom{B}{A}$ contains all substructures A' of B that are isomorphic to A.

Theorem 6.4
For all $r, t_A, t_B \in \mathbb{N}$ and all $a_1, \dots a_{t_A}, b \in \mathbb{N}$ there exist $t_C, c \in \mathbb{N}$ such that the following holds: If

- $A = \{(x, j) : x \in \{1, \dots, a_j\}, 0 < j \leq t_A\}$
- $B = \{(x, j) : x \in \{1, \dots, b\}, 0 < j \leq t_B\}$
- $C = \{(x, j) : x \in \{1, \dots, c\}, 0 < j \leq t_C\}$
- *we r-color $\binom{C}{A}$,*

then there is a monochromatic copy of B in C.

Proof:
For simplicity we assume $a_1 = \dots = a_{t_A} = a$ (but with a minor modification of the product Ramsey theorem the following proof also works for distinct values for a_i).

Set $t(C) = R(t(A), t(B), r)$ and $\gamma = \binom{t(C)}{t(A)}$. Moreover, define $c(1) = PR(r, a, t(A), b)$ and $c(i+1) = PR(r, a, t(A), c(i))$ for $i > 1$ and let $c = c(\gamma)$. Let

- $A = A_1 \cup \dots \cup A_{t(A)}$ with $A_i = \{(x, i) : x \in \{1, \dots, a\}\}$ for $0 < i \leq t(A)$,
- $B = B_1 \cup \dots \cup B_{t(B)}$ with $B_i = \{(x, i) : x \in \{1, \dots, b\}\}$ for $0 < i \leq t(B)$,
- $C = C_1 \cup \dots \cup C_{t(C)}$ with $C_i = \{(x, i) : x \in \{1, \dots, c\}\}$ for $0 < i \leq t(C)$,

and consider a coloring χ of $\binom{C}{A}$.

We establish the theorem by finding sets $\tilde{C}_j \subset C_j$ of cardinality b for $1 \leq j \leq t_C$ such that $\tilde{C} = \tilde{C}_1 \cup \ldots \cup \tilde{C}_j$ has the following property: For all $i_1, \dots, i_{t_A} \in \{1, \dots, t_C\}$ all copies of A in $\tilde{C_{i_1}} \cup \ldots \cup \tilde{C_{i_{t_A}}}$ have the same color. This uniquely determines an r-coloring of the t_A-element subsets of the integers $1, \dots, t_C$. By the choice of t_C and the finite Ramsey theorem we find a monochromatic t_B-element subset in these integers and accordingly a monochromatic copy of B in C.

For finding the sets $\tilde{C}_j$ we repeatedly apply the product Ramsey theorem. The idea is to do the following steps for each choice of $i_1, \dots, i_{t_A} \in \{1, \dots, t_C\}$: Consider $C^* = C_{i_1} \cup \ldots \cup C_{i_{t_A}}$ and choose the largest monochromatic subset (with respect to χ) of C^*. For the following steps, restrict each set C_{i_j} to $C_{i_j} \cap C^*$. Then continue with the next choice of $i_1, \dots, i_{t_A}$. By the choice of c and the product Ramsey theorem we will get subsets $\tilde{C}_j$ of C_j with $|\tilde{C}_j| \geq b$ at the end of this procedure. We omit the details. □

This immediately implies the desired result.

Corollary 6.5
The class of increasing sequences of decreasing sequences and the class of decreasing sequences of increasing sequences are Ramsey classes.

6.2 The Class of all Permutations

We believe that also the class of all permutations forms a Ramsey class. Although not all details are worked out yet, we will outline how this can possibly been proven. For this we use the amalgamation technique, which was introduced by Nešetřil and Rödl (cf. Nešetřil, 1995). This technique consists of two main parts: the *partite lemma* and the *partite construction*. We start with some definitions.

Definition: *Let $a \in \mathbb{N}$. An* a-partite *permutation $P = (X_1 \cup \ldots \cup X_a, <_1, <_2)$ is a permutation on the union of disjoint sets $X_1, \ldots, X_a$ such that $x <_1 y$ for $x \in X_i$ and $y \in X_j$ whenever $i < j$ and $x <_1 y$ iff $x <_2 y$ for $x, y \in X_i$. The sets X_i are called the* parts *of P. An a-partite permutation is a* transversal *if each part is of size one. As subpermutation of P we now only consider a-partite subpermutations $P' = (X'_1 \cup \ldots \cup X'_a, <'_1, <'_2)$ (where the X'_i are possibly empty).*

The partite lemma asserts the following.

Lemma 6.6
Let A be an a-partite transversal and B an arbitrary a-partite permutation. Then for any $r \in \mathbb{N}$ there exists an a-partite permutation C such that $C \to (B)^A_r$.

We hope to prove this lemma along the lines of the proof of the partite lemma for relational structures as outlined in Nešetřil, 1995.

For the partite construction we additionally need the following lemma.

Lemma 6.7
For each $a \in \mathbb{N}$ the class of a-partite permutations has the amalgamation property.

This can easily be seen by a direct construction.

In the partite construction we then inductively define permutations $P_0, \ldots, P_{\binom{q}{b}}$ where $q = R(a, b, r)$. Let A and B be arbitrary permutations of sizes a and b, respectively. We consider A as an a-partite permutation and B as a b-partite permutation. Let P_0 be a q-partite permutation contaning $\binom{q}{b}$ disjoint subpermutations isomorphic to B such that each choice of b parts of P_0 induces a copy of B. Such a permutation can easily be constructed. P_{i+1} is then obtained from P_i by choosing b parts (that have not been chosen before) of the q possible parts and performing the following construction: Let D_i be the permutation induced on these b parts. By the partite lemma there is a b-partite permutation E_i such that $E_i \to (D_i)^A_r$. Construct P_{i+1} by amalgamating copies of P_i along the copies of B in E_i. The permutation $P_{\binom{q}{b}}$ then has the desired properties. For this we can proceed similar to the proof of theorem 6.4 by backward induction: In P_{i+1} choose a copy of P_i such that all copies of A in D_i have the same color. This copy exists by the partite lemma. Finally, we can apply the Ramsey theorem to the copy of P_0 we obtained in this way, for finding a monochromatic copy of B.

This (if all the details can be completed) shows that there is a permutation C such that any r coloring of $\binom{C}{A}$ contains a monochromatic copy of B and therefore the class of all permutations form a Ramsey class.

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