# EMBEDDING INTO BIPARTITE GRAPHS

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Abstract. The conjecture of Bollobás and Komlós, recently proved by Böttcher, Schacht, and Taraz [Math. Ann. 343(1), 175–205, 2009], implies that for any $\gamma > 0$, every balanced bipartite graph on $2n$ vertices with bounded degree and sublinear bandwidth appears as a subgraph of any $2n$-vertex graph $G$ with minimum degree $(1+\gamma)n$, provided that $n$ is sufficiently large. We show that this threshold can be cut in half to an essentially best-possible minimum degree of $(\frac{1}{2}+\gamma)n$ when we have the additional structural information of the host graph $G$ being balanced bipartite.

This complements results of Zhao [to appear in SIAM J. Discrete Math.], as well as Hladký and Schacht [to appear in SIAM J. Discrete Math.], who determined a corresponding minimum degree threshold for $K_{r,s}$-factors, with $r$ and $s$ fixed. Moreover, it implies that the set of Hamilton cycles of $G$ is a generating system for its cycle space.

*Keywords:* Graph theory (05Cxx), Extremal combinatorics (05Dxx), Graph embedding

## 1. Introduction

The Bollobás–Komlós conjecture, recently proved in [5], provides a sufficient and essentially best possible minimum degree condition for the containment of $r$-chromatic spanning graphs $H$ of bounded maximum degree and small bandwidth. Here, a graph is said to have bandwidth at most $b$, if there exists a labelling of the vertices by numbers $1, \ldots, n$, such that for every edge $\{i, j\}$ of the graph we have $|i-j| \leq b$.

**Theorem 1** (Böttcher, Schacht, Taraz [5]). *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. If $H$ is an $r$-chromatic graph on $n$ vertices with $\Delta(H) \leq \Delta$ and bandwidth at most $\beta n$ and if $G$ is a graph on $n$ vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r}+\gamma)n$, then $G$ contains a copy of $H$.* □

This theorem in particular implies that for any $\gamma > 0$, every bipartite graph $H$ on $2n$ vertices with bounded degree and sublinear bandwidth appears as a subgraph of any $2n$-vertex graph $G$ with minimum degree $(1+\gamma)n$, provided that $n$ is sufficiently large. This bound is essentially best possible for an almost trivial reason: there are graphs $G$ with minimum degree just slightly below $n$ that are not connected. Such $G$ clearly do not contain a connected $H$ as a subgraph. These graphs are simply too different in structure from $H$.

One may ask, however, whether it is possible to lower the minimum degree threshold in Theorem 1 for graphs $G$ and $H$ that are structurally more similar and, in particular, have the same chromatic number. In this paper we will pursue

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this question for the case of balanced bipartite graphs, i.e., bipartite graphs on $2n$ vertices with $n$ vertices in each colour class.

Dirac's theorem [7] implies that a $2n$-vertex graph $G$ with minimum degree at least $n$ contains a Hamilton cycle. If $G$ is balanced bipartite, it follows from a theorem of Moon and Moser [17] that this minimum degree threshold can be cut almost in half.

**Theorem 2.** *Let $G$ be a balanced bipartite graph on $2n$ vertices. If $\delta(G) \geq \frac{n}{2} + 1$, then $G$ contains a Hamilton cycle.*

We prove that slightly increasing this minimum degree bound suffices to obtain all balanced bipartite graphs with bounded maximum degree and sublinear bandwidth as subgraphs, and thereby establishing the following bipartite analogue of Theorem 1, halving the minimum degree threshold in that result.

**Theorem 3.** *For all $\gamma$ and $\Delta$ there is a positive constant $\beta$ and an integer $n_0$ such that for all $n \geq n_0$ the following holds. Let $G$ and $H$ be balanced bipartite graphs on $2n$ vertices such that $G$ has minimum degree $\delta(G) \geq (\frac{1}{2} + \gamma)n$ and $H$ has maximum degree $\Delta$ and bandwidth at most $\beta n$. Then $G$ contains a copy of $H$.*

Results of a similar nature have recently been established by Zhao [19], and by Hladký and Schacht [10] who considered the special case of coverings of $G$ with disjoint copies of complete bipartite graphs. Moreover, as a first step towards Theorem 3, in [8] this result was proved for a special balanced bipartite connected graph (the so-called Möbius ladder).

We remark that the bandwidth condition in Theorem 3 cannot be omitted. Indeed, Abbasi [1] proved that the assertion of Theorem 1 gets false if $\beta > 4\gamma$. The graph $H$ he constructs for this purpose is a balanced bipartite graph and it is not difficult to see that Abbasi's host graph contains a bipartite subgraph meeting our conditions but not containing $H$. However, the bound on $\beta$ coming from our proof is very small, having a tower-type dependence on $1/\gamma$.

The proof of Theorem 3 is given in Section 3. It is based on Szemerédi's regularity lemma which we introduce in the following section. In Sections 4 and 5 we provide the proofs of the remaining lemmas that are used in the proof of Theorem 3.

## 2. The regularity method

In this section we formulate a version of Szemerédi's regularity lemma [18] that is convenient for our application (Lemma 5), introduce all necessary definitions, and formulate an embedding lemma for spanning subgraphs (Lemma 7).

The regularity lemma relies on the concept of a regular pair. To define this, let $G = (V, E)$ be a graph and $0 \leq \varepsilon, d \leq 1$. For disjoint nonempty vertex sets $U, W \subseteq V$ the *density* $d(U, W)$ of the pair $(U, W)$ is the number of edges that run between $U$ and $W$ divided by $|U||W|$. A pair $(U, W)$ with density at least $d$ is $(\varepsilon, d)$-*regular* if $|d(U', W') - d(U, W)| \leq \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$. The following useful property of regular pairs follows immediately from the definition.

**Proposition 4.** *Let $G = (A, B)$ be an $(\varepsilon, d)$-regular pair. Let $B'$ be a subset of $B$ with $|B'| \geq \varepsilon|B|$. Then there are at most $\varepsilon|A|$ vertices in $A$ with less than $(d-\varepsilon)|B'|$ neighbours in $B'$.* $\square$

The regularity lemma asserts that each graph admits a partition into relatively few vertex classes of equal size such that most pairs of these classes form an $\varepsilon$-regular pair. The following definition makes this precise. A partition $V_0\dot\cup V_1\dot\cup\cdots\dot\cup V_k$ of $V$ with $|V_0|\le\varepsilon|V|$ is *$(\varepsilon,d)$-regular on* a graph $R=([k],E_R)$ if $ij\in E_R$ implies that $(V_i,V_j)$ is an $(\varepsilon,d)$-regular pair in $G$. If such a partition exists, we also say that $R$ is an *$(\varepsilon,d)$-reduced graph* of $G$. Moreover, $R$ is the *maximal* $(\varepsilon,d)$-reduced graph of the partition $V_0\dot\cup V_1\dot\cup\cdots\dot\cup V_k$ if there is no $ij\notin E_R$ with $i,j\in[k]$ such that $(V_i,V_j)$ is $(\varepsilon,d)$-regular. A partition $V_0\dot\cup V_1\dot\cup\cdots\dot\cup V_k$ of $V$ is an *equipartition* if $|V_i|=|V_j|$ for all $i,j\in[k]$. The partition classes $V_i$ with $i\in[k]$ are also called *clusters* of $G$ and $V_0$ is the *exceptional set.* When the exceptional set $V_0$ is empty (or when we want to ignore it as well as its size) then we may omit it and say that $V_1\dot\cup\dots\dot\cup V_k$ is regular on $R$. An $(\varepsilon,d)$-regular pair $(U,W)$ is *$(\varepsilon,d)$-super-regular* if every vertex $u\in U$ has degree $\deg_W(u)\ge d|W|$ and every $w\in W$ has $\deg_U(w)\ge d|U|$. For a graph $G=(V,E)$ a partition $V=V_0\dot\cup V_1\dot\cup\cdots\dot\cup V_k$ is said to be *super-regular on a graph* $R$ with vertex set $V_R$, $V_R\subseteq[k]$, if $(V_i,V_j)$ is super-regular whenever $ij$ is an edge of $R$.

In this paper we consider bipartite graphs and the regular partitions that appear in the proof of Theorem 3 refine some bipartition and their reduced graphs are bipartite. More precisely, for a bipartite graph $G=(A\dot\cup B,E)$ we will obtain a partition $(A_0\dot\cup B_0)\dot\cup A_1\dot\cup B_1\dot\cup\dots\dot\cup A_k\dot\cup B_k$ that is $(\varepsilon,d)$-regular (or super-regular) on some bipartite graph $R$ such that $A=A_0\dot\cup\dots\dot\cup A_k$ and $B=B_0\dot\cup\dots\dot\cup B_k$. In particular we have two different exceptional sets now, one in $A$ called $A_0$ and one in $B$ called $B_0$, each of size $\varepsilon n$ at most. Such a partition is an equipartition if $|A_1|=|B_1|=|A_2|=\cdots=|A_k|=|B_k|$. In addition, we consider only regular pairs running between the bipartition classes, i.e., pairs of the form $(A_i,B_j)$. Consequently, all reduced graphs (also the maximal reduced graph of a partition) are bipartite.

We now state the version of the regularity lemma that we will use. This is a corollary of the degree form of the regularity lemma (see, e.g., [14, Theorem 1.10]) and is tailored for embedding applications in balanced bipartite graphs satisfying some minimum degree condition. We sketch its proof below.

**Lemma 5** (regular partitions of bipartite graphs)**.** *For every $\varepsilon'>0$ and for every $\Delta,k_0\in\mathbb{N}$ there exists $K_0=K_0(\varepsilon',k_0)\in\mathbb{N}$ such that for every $0\le d'\le1$, for*

$$\varepsilon'':=\frac{2\Delta\varepsilon'}{1-\varepsilon'\Delta}\qquad\text{and}\qquad d'':=d'-2\varepsilon'\Delta\,,$$

*and for every bipartite graph $G=(A\dot\cup B,E)$ with $|A|=|B|\ge K_0$ and $\delta(G)\ge\nu|G|$ for some $0<\nu<1$ there exists a graph $R$ and an integer $k$ with $k_0\le k\le K_0$ with the following properties:*

(*a*) *$R$ is an $(\varepsilon',d')$-reduced graph of an equipartition of $G$ and $|V(R)|=2k$.*

(*b*) *$\delta(R)\ge(\nu-d'-\varepsilon'')|R|$.*

(*c*) *For every subgraph $R^*\subseteq R$ with $\Delta(R^*)\le\Delta$ there is an equipartition*

$$A\dot\cup B=A''_0\dot\cup B''_0\dot\cup A''_1\dot\cup B''_1\dot\cup\dots\dot\cup A''_k\dot\cup B''_k$$

*with $A''_i\subseteq A$ and $B''_i\subseteq B$ for all $0\le i\le k$ and $(\varepsilon'',d'')$-reduced graph $R$, which in addition is $(\varepsilon'',d'')$-super-regular on $R^*$.*

*Proof (sketch).* The proof of this lemma is a standard combination of three standard tools. As a first step we simulate the proof of the degree-form (see [13], Lemma 2.1, or the survey [14]) of the regularity lemma starting with $A\dot\cup B$ as the initial partition (see also [6, Chapter 7.4]). This yields a partition into clusters $A_0, \dots, B_k$ such that for all vertices $v \notin A_0 \cup B_0$ there are at most $(d' + \varepsilon')n$ edges $e \in E$ with $v \in e$ such that $e$ is not in some $(\varepsilon', d')$-regular pair $(A_i, B_j)$. Hence we get (*a*). Let $R$ be the maximal (bipartite) $(\varepsilon', d')$-reduced graph of this partition. Then it is easy to see that $R$ inherits the minimum degree condition of $G$ (except for a small loss), see [15, Proposition 9]. This yields (*b*). Finally, for all pairs $(A_i, B_j)$ with $i, j \in [k]$ that correspond to edges in $R^*$ we take those vertices in $A_i$ or $B_i$ that have too few edges in $(A_i, B_j)$ and move them to $A_0$ or $B_0$, respectively. See [15, Proposition 8] for details. This yields (*c*). □

2.1. **Embedding into regular partitions.** For embedding *spanning subgraphs* $H$ into graphs $G$ with high minimum degree the blow-up lemma of Komlós, Sárközy and Szemerédi [12] has proved to be an extremely valuable tool. The blow-up lemma guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently super-regular pairs. In fact, this lemma is more general and allows the embedding of graphs $H$ into partitions that are super-regular on some graph $R$ if there is a homomorphism from $H$ to $R$ that does not send too many vertices of $H$ to each cluster of $R$.

When embedding a spanning graph $H$ into a host graph $G$ a well-established strategy is to utilise the blow-up lemma on small super-regular "spots" in a regular partition of $G$ for embedding most of the vertices of $H$, and to use a greedy embedding method to embed the few other vertices first. This embedding method is summarised in the next lemma, the general embedding lemma. Before stating it we need to identify conditions under which it is possible to proceed in the way just described. This is addressed in the following definition that specifies when a partition of $H$ is "compatible" with a regular partition of $G$ with reduced graph $R$ and a subgraph $R'$ of $R$ such that edges of $R'$ correspond to dense super-regular pairs. In this definition we require that the partition of $H$ has smaller partition classes than the partition of $G$ (condition (*i*)), and that edges of $H$ run only between partition classes that correspond to a dense regular pair in $G$ (condition (*ii*)). Further, in each partition class $W_i$ of $H$ we identify two subsets $S_i$ and $T_i$ that are both supposed to be small (condition (*iii*)). The set $S_i$ contains those vertices that send edges over pairs that do not belong to the super-regular pairs specified by $R'$ and $T_i$ contains neighbours of such vertices.

**Definition 6** ($\varepsilon$-compatible)**.** *Let $H = (W, E_H)$ and $R = ([k], E_R)$ be graphs and let $R' = ([k], E_{R'})$ be a subgraph of $R$. We say that a vertex partition $W = (W_i)_{i\in[k]}$ of $H$ is* $\varepsilon$-compatible *with an integer partition $(n_i)_{i\in[k]}$ of $n$ and with $R' \subseteq R$ if the following holds. For $i \in [k]$ let $S_i$ be the set of vertices in $W_i$ with neighbours in some $W_j$ with $ij \notin E_{R'}$ and $i \neq j$, set $S := \bigcup S_i$ and $T_i := N_H(S) \cap (W_i \setminus S)$. Then for all $i, j \in [k]$ we have that*

(*i*) $|W_i| \le n_i$,
(*ii*) $xy \in E_H$ *for* $x \in W_i$ *and* $y \in W_j$ *implies* $ij \in E_R$,
(*iii*) $|S_i| \le \varepsilon n_i$ *and* $|T_i| \le \varepsilon \cdot \min\{n_j : i \text{ and } j \text{ are in the same component of } R'\}$.

*The partition $W = (W_i)_{i\in[k]}$ of $H$ is $\varepsilon$-compatible with a partition $V = (V_i)_{i\in[k]}$ of a graph $G$ and with $R' \subseteq R$ if $W = (W_i)_{i\in[k]}$ is $\varepsilon$-compatible with $(|V_i|)_{i\in[k]}$ and with $R' \subseteq R$.*

The general embedding lemma asserts that a bounded-degree graph $H$ can be embedded into a graph $G$ if $H$ and $G$ have compatible partitions. A proof can be found in [3, Section 3.3.3].

**Lemma 7** (general embedding lemma)**.** *For all $d, \Delta, r > 0$ there is a constant $\varepsilon = \varepsilon(d, \Delta, r) > 0$ such that the following holds. Let $G = (V, E)$ be an $n$-vertex graph that has a partition $V = (V_i)_{i \in [k]}$ with $(\varepsilon, d)$-reduced graph $R$ on $[k]$ which is $(\varepsilon, d)$-super-regular on a graph $R' \subseteq R$ with connected components having at most $r$ vertices each. Further, let $H = (W, E_H)$ be an $n$-vertex graph with maximum degree $\Delta(H) \le \Delta$ that has a vertex partition $W = (W_i)_{i \in [k]}$ which is $\varepsilon$-compatible with $V = (V_i)_{i \in [k]}$ and $R' \subseteq R$. Then $H \subseteq G$.* $\square$

For applying the general embedding lemma to *spanning* graphs $H$ we need a partition of the graph $H$ whose partition classes match the sizes of a regular partition of $G$ *precisely*. However, usually we cannot guarantee that this is the case for a regular partition obtained from Lemma 5. Hence it will become necessary to modify such a regular partition slightly by moving some vertices into different clusters. The following lemma asserts that the resulting partition is still regular with somewhat worse parameters. For a proof see [4, Proposition 8].

**Proposition 8.** *Let $(A, B)$ be an $(\varepsilon, d)$-regular pair and let $\hat{A}$ and $\hat{B}$ be vertex sets with $|\hat{A} \triangle A| \le \alpha |\hat{A}|$ and $|\hat{B} \triangle B| \le \beta |\hat{B}|$. Then $(\hat{A}, \hat{B})$ is an $(\hat{\varepsilon}, \hat{d})$-regular pair where*

$$\hat{\varepsilon} := \varepsilon + 3(\sqrt{\alpha} + \sqrt{\beta}) \qquad \text{and} \qquad \hat{d} := d - 2(\alpha + \beta).$$

*If, moreover, $(A, B)$ is $(\varepsilon, d)$-super-regular and each vertex $v$ in $\hat{A}$ has at least $d|\hat{B}|$ neighbours in $\hat{B}$ and each vertex $v$ in $\hat{B}$ has at least $d|\hat{A}|$ neighbours in $\hat{A}$, then $(\hat{A}, \hat{B})$ is $(\hat{\varepsilon}, \hat{d})$-super-regular with $\hat{\varepsilon}$ and $\hat{d}$ as above.* $\square$

## 3. The proof of the main theorem

In the proof of Theorem 3 we will use the general embedding lemma (Lemma 7). For applying this lemma we need compatible partitions of the graphs $G$ and $H$ which are provided by the next two lemmas. We start with the lemma for $G$ which constructs a regular partition of $G$ whose reduced graph $R$ contains a perfect matching within a Hamilton cycle of $R$. The lemma guarantees, moreover, that the regular partition is super-regular on this perfect matching (see Figure 1) and that the cluster sizes in the partition can be slightly changed.

We remark that, throughout, $A \dot\cup B$ will denote the vertex set of the host graph $G$ while $X \dot\cup Y$ is the vertex set of the bipartite graph $H$ we would like to embed. The sets $A_i$ and $B_i$ with $i \in [k]$ for some integer $k$ will denote the clusters of a regular partition of $G$ as well as for the vertices of a corresponding reduced graph.

**Lemma 9** (Lemma for $G$)**.** *For every $\gamma > 0$ there exists $d_{\mathrm{LG}} > 0$ such that for every $\varepsilon > 0$ and every $k_0 \in \mathbb{N}$ there exist $K_0 \in \mathbb{N}$ and $\xi_{\mathrm{LG}} > 0$ with the following properties: For every $n \ge K_0$ and for every balanced bipartite graph $G = (A \dot\cup B, E)$ on $2n$ vertices with $\delta(G) \ge \big(1/2 + \gamma\big)n$ there exists $k_0 \le k \le K_0$ and a partition $(n_i)_{i \in [k]}$ of $n$ with $n_i \ge n/(2k)$ such that for every partition $(a_i)_{i \in [k]}$ of $n$ and $(b_i)_{i \in [k]}$ of $n$ satisfying $a_i \le n_i + \xi_{\mathrm{LG}} n$ and $b_i \le n_i + \xi_{\mathrm{LG}} n$, for all $i \in [k]$, there exist partitions*

$$A = A_1 \dot\cup \cdots \dot\cup A_k \quad \text{and} \quad B = B_1 \dot\cup \cdots \dot\cup B_k$$

*such that*

(G1) $|A_i| = a_i$ *and* $|B_i| = b_i$ *for all* $i \in [k]$,
(G2) $(A_i, B_i)$ *is* $(\varepsilon, d_{\text{LG}})$*-super-regular for every* $i \in [k]$.
(G3) $(A_i, B_{i+1})$ *is* $(\varepsilon, d_{\text{LG}})$*-regular for every* $i \in [k]$.

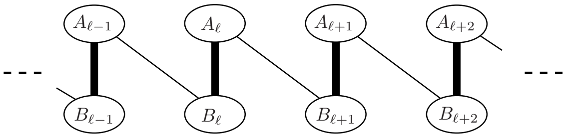

FIGURE 1. The regular partition constructed by Lemma 9 with super-regular pairs $(A_i, B_i)$ and regular pairs $(A_i, B_{i+1})$.

The proof of this lemma is presented in Section 4. The following lemma, which we will prove in Section 5, constructs the corresponding partition of $H$. It guarantees that the $2k$ partition classes of $H$ are roughly of the same sizes as the corresponding partition classes of $G$ (see (H3)), and that all edges of $H$ are mapped to edges of a cycle $C$ on $2k$ vertices and all edges except those incident to a very small set $S$ (see (H1)) are in fact mapped to the edges of a perfect matching in $C$ (see (H2)).

**Lemma 10** (Lemma for $H$)**.** *For every* $k \in \mathbb{N}$ *and every* $\xi > 0$ *there exists* $\beta > 0$ *and* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *and for every balanced bipartite graph* $H = (X\dot\cup Y, F)$ *on* $2n$ *vertices having* $\text{bw}(H) \leq \beta n$ *and for every integer partition* $n = n_1 + \cdots + n_k$ *with* $n_i \leq n/8$ *there exists a set* $S \subseteq V(H)$ *and a graph homomorphism* $f\colon V(H) \to V(C)$, *where* $C$ *is the cycle on the vertices* $A_1, B_2, A_2, \ldots, B_k, A_k, B_1, A_1$, *such that*

(H1) $|S| \leq \xi \cdot 2k \cdot n$,
(H2) *for every* $\{x, y\} \in F$ *with* $x \in X \setminus S$ *and* $y \in Y \setminus S$ *there is* $i \in [k]$ *such that* $f(x) \in A_i$ *and* $f(y) \in B_i$,
(H3) $|f^{-1}(A_i)| < n_i + \xi n$ *and* $|f^{-1}(B_i)| < n_i + \xi n$ *for every* $i \in [k]$.

With Lemmas 7 (the general embedding lemma), Lemma 9 (the lemma for $G$) and Lemma 10 (the lemma for $H$) at our disposal, we are ready to give the proof of the main theorem.

*Proof of Theorem 3.* Given $\gamma$ and $\Delta$, let $d$ be the constant provided by Lemma 9 for input $\gamma$. Let $\varepsilon$ be the constant Lemma 7 returns for input $d$, $\Delta$, and $r = 2$. We continue the application of Lemma 9 with input $\varepsilon$ and $k_0 := 2$ and get constants $K_0$ and $\xi_{\text{LG}}$ and set $\xi_{\text{LH}} := \xi_{\text{LG}}\varepsilon/(100\Delta K_0^2)$. Further let $\beta$ be the minimum of all the values $\beta_k$ and $n_0'$ be the maximum of all the values $n_0^{(k)}$ that Lemma 10 returns for input $k$ and $\xi$ where $k$ runs from $k_0$ to $K_0$. Finally, we set $n_0 := \max\{n_0', K_0\}$.

Let $G = (A\dot\cup B, E)$ and $H = (X\dot\cup Y, F)$ be balanced bipartite graphs on $2n$ vertices with $n \geq n_0$, $\delta(G) \geq (\frac{1}{2} + \gamma)n$, $\Delta(H) \leq \Delta$, and $\text{bw}(H) \leq \beta n$. We apply Lemma 9 to the graph $G$ in order to obtain an integer $k$ and an integer partition $(n_i)_{i\in[k]}$ with $n_i \geq \frac{1}{2}n/k$ for all $i \in [k]$. Next, we apply Lemma 10 to the graph $H$ and the integer partition $(n_i)_{i\in[k]}$ and get a vertex set $S \subseteq X \cup Y$ and a homomorphism $f$ from $H$ to the cycle $C$ on vertices $A_1, B_2, A_2, \ldots B_k, A_k, B_1, A_1$ such that (H1)–(H3) are satisfied. With this we can define the integer partitions $(a_i)_{i\in[k]}$ and $(b_i)_{i\in[k]}$ required for the continuation of Lemma 9: set $a_i := |f^{-1}(A_i)|$ and $b_i := |f^{-1}(B_i)|$ for all $i \in [k]$. By (H3) we have $a_i \leq n_i + \xi_{\text{LH}}n \leq n_i + \xi_{\text{LG}}n$

and $b_i \le n_i + \xi_{\text{LG}} n$ for all $i \in [k]$. It follows that Lemma 9 now gives us vertex partitions $A = (A_i)_{i\in[k]}$ and $B = (B_i)_{i\in[k]}$ for $G$ such that (G1)–(G3) hold. We complement this with vertex partitions $X = (X_i)_{i\in[k]}$ and $Y = (Y_i)_{i\in[k]}$ for $H$ defined by $X_i := f^{-1}(A_i)$ and $Y_i := f^{-1}(B_i)$ and claim that we can use the general embedding lemma (Lemma 7) for these vertex partitions of $G$ and $H$.

Indeed, first observe that (G2) and (G3) imply that the partition $V(G) = (A_i)_{i\in[k]}\dot\cup(B_i)_{i\in[k]}$ is $(\varepsilon, d)$-regular on the graph $C$. Further, by (G3) this partition is $(\varepsilon, d)$-super-regular on the graph $R'$ on the same vertices as $C$ and with edges $A_iB_i$ for all $i \in [k]$. Notice that the components of $R'$ have size $r = 2$. It follows that we can apply Lemma 7 if the vertex partition $V(H) = (X_i)_{i\in[k]}\dot\cup(Y_i)_{i\in[k]}$ is $\varepsilon$-compatible with the partition $V(G) = (A_i)_{i\in[k]}\dot\cup(B_i)_{i\in[k]}$ and with $R' \subseteq C$. To check this first note that by (G1) we have $|A_i| = a_i = |X_i|$ and $|B_i| = b_i = |Y_i|$ for all $i \in [k]$ and thus Property *(i)* of an $\varepsilon$-compatible partition is satisfied. Since $f$ is a homomorphism from $H$ to $C$ we also immediately get Property *(ii)* for $(X_i)_{i\in[k]}\dot\cup(Y_i)_{i\in[k]}$. In addition, since $|A_i| = a_i \le n_i + \xi_{\text{LH}} n$ for all $i \in [k]$, we also have $|A_i| \ge n_i - k\xi_{\text{LH}} n \ge \frac{1}{2}n/k - k\xi_{\text{LH}} n \ge \Delta\xi_{\text{LH}} 2kn/\varepsilon$ by the choice of $\xi_{\text{LH}}$. This together with (H1) implies that $|S \cap A_i| \le \xi_{\text{LH}} 2kn \le \varepsilon|A_i|$ and $|N_H(S) \cap A_i| \le \Delta|S| \le \Delta\xi_{\text{LH}} 2kn \le \varepsilon|A_j|$ for all $i, j \in [k]$. Similarly we get $|S \cap B_i| \le \varepsilon|B_i|$ and $|N_H(S) \cap B_i| \le \varepsilon|B_j|$ for all $i, j \in [k]$. This clearly implies Property *(iii)* of an $\varepsilon$-compatible partition.

Accordingly we can apply Lemma 7 to the graphs $G$ and $H$ with their partitions $V(G) = (A_i)_{i\in[k]}\dot\cup(B_i)_{i\in[k]}$ and $V(H) = (X_i)_{i\in[k]}\dot\cup(Y_i)_{i\in[k]}$, respectively, which implies that $H$ is a subgraph of $G$. □

## 4. A REGULAR PARTITION OF $G$ WITH A SPANNING CYCLE

In this section we will prove the Lemma for $G$. This lemma is a consequence of the regularity lemma (Lemma 5), Theorem 2, and the following lemma which states that, under certain circumstances, we can adjust a (super)-regular partition in order to meet a request for slightly differing cluster sizes.

**Lemma 11.** *Let $k \ge 1$ be an integer, $0 < \xi \le 1/(20k^2)$ and let $G = (A\dot\cup B, E)$ be a balanced bipartite graph on $2n$ vertices with partitions $A = A'_1\dot\cup\cdots\dot\cup A'_k$ and $B = B'_1\dot\cup\cdots\dot\cup B'_k$ such that $|A'_i|, |B'_i| \ge n/(2k)$ and $(A'_i, B'_i)$ is $(\varepsilon', d')$-super-regular and $(A'_i, B'_{i+1})$ is $(\varepsilon', d')$-regular for all $i \in [k]$. Let $(a'_i)_{i\in[k]}$ and $(b'_i)_{i\in[k]}$ be integers such that $a'_i, b'_i \le \xi n$ for all $i \in [k]$ and $\sum_{i\in[k]} a'_i = \sum_{i\in[k]} b'_i = 0$. Then there are partitions $A = A_1\dot\cup\cdots\dot\cup A_k$ and $B = B_1\dot\cup\cdots\dot\cup B_k$ with $|A_i| = |A'_i| + a'_i$ and $|B_i| = |B'_i| + b'_i$ and such that $(A_i, B_i)$ is $(\varepsilon, d)$-super-regular and $(A_i, B_{i+1})$ is $(\varepsilon, d)$-regular for all $i \in [k]$ where $\varepsilon := \varepsilon' + 100k\sqrt{\xi}$ and $d := d' - 100k^2\sqrt{\xi} - \varepsilon'$.*

*Proof.* The lemma will be proved by performing a simple redistribution algorithm that will iteratively adjust the cluster sizes. Throughout the process, we denote by $A_i$ and $B_i$ the changing clusters, beginning with $A_i := A'_i$ and $B_i := B'_i$. We call $A_i$ a *sink* when $|A_i| < |A'_i| + a'_i$, and a *source* when $|A_i| > |A'_i| + a'_i$, and analogously for $B'_i$. Each iteration of the algorithm will have the effect that the number of vertices in a single source decreases by one, the number of vertices in a single sink increases by one, and all other cluster cardinalities stay the same.

We start by describing one iteration of the algorithm. Obviously, as long as not every cluster in $A$ has exactly the desired size, there is at least one source. We choose an arbitrary source $A_i$, and, as will be further explained below, the regularity of the pair $(A_i, B_{i+1})$ implies that within $A_i$ there is a large set of vertices each of which can be added to the neighbouring cluster $A_{i+1}$ while preserving the super-regularity of the pair $(A_{i+1}, B_{i+1})$. We do this with one arbitrary vertex from this set. Thereafter, within $A_{i+1}$ there is again a large set of vertices (the newly arrived vertex may or may not be one of them) suitable for being moved into $A_{i+2}$ while preserving the super-regularity of the pair $(A_{i+2}, B_{i+2})$, and we again do this with one arbitrary vertex from this set. We then continue in this way until for the first time we move a vertex into a sink. (It may happen that it is not the vertex we initially took out of $A_i$ that arrives in the sink.) This is the end of the iteration.

We repeat such iterations as long as there are sources, i.e. we choose an arbitrary source and repeat what we have just described. Since each iteration ends with adding a vertex to a sink while not changing the cardinality of the clusters visited along the way, we do not increase the number of vertices in any source, let alone create a new source, and hence after a finite number of iterations (which we will estimate below) the algorithm ends with no sources remaining and therefore all clusters within $A$ having exactly the desired size.

We then repeat what we have just described for the clusters within $B$, the only difference being that vertices get moved from $B_i$ into $B_{i-1}$, not $B_{i+1}$, since only in this direction a regular pair can be used ($(A_{i-1}, B_i)$ is regular, $(A_{i+1}, B_i)$ need not be regular).

We now analyse the algorithm quantitatively. Clearly, the total number of iterations (we call it $t$) is at most the sum of all positive $a'_i$ and all positive $b'_i$. Obviously, both the sum of all positive $a'_i$ and the sum of all positive $b'_i$ is bounded from above by $\frac{1}{2}k\xi n$, hence

$$t \leq \tfrac{1}{2}k\xi n + \tfrac{1}{2}k\xi n = k\xi n. \tag{1}$$

We will now use this bound together with Proposition 8 to estimate the effect of the redistribution on the regularity and density parameters. Since in each iteration each cluster receives at most one vertex and loses at most one vertex, for every $i \in [k]$ and after any step of the algorithm, we have

$$|A_i \Delta A'_i| \leq 2t \leq 2k\xi n\,,$$

and analogously $|B_i \Delta B'_i| \leq 2k\xi n$. We now invoke Proposition 8 on the pairs $(A_i, B_i)$ and $(A_i, B_{i+1})$, once with $\hat{A} := A_i$, $\hat{B} := B_i$ then with $\hat{A} := A_i$, $\hat{B} := B_{i+1}$ and we claim that we may use $\alpha := \beta := 16k^2\xi$. Indeed, we have $|A_i| \geq |A'_i| - t \geq n/(2k) - 2k\xi n$ and because $\xi \leq 1/(20k^2)$ implies $2k\xi n \leq 5k\xi n - 20k^3\xi^2 n$, hence $|A_i \Delta A'_i| \leq 2k\xi n \leq (5k\xi - 20k^3\xi^2)n = 10k^2\xi(n/(2k) - 2k\xi n) \leq \alpha|A'_i|$, and analogously $|B_i \Delta B'_i| \leq \beta|B'_i|$. By Proposition 8, every pair $(A_i, B_i)$ and $(A_i, B_{i+1})$ is $(\hat{\varepsilon}, \hat{d})$-regular with $\hat{\varepsilon} := \varepsilon' + 24k\sqrt{\xi}$ and $\hat{d} := d' - 64k^2\xi$, hence $\hat{\varepsilon} \leq \varepsilon$ and $\hat{d} \geq d$, proving the parameters claimed in the lemma, as far as mere regularity goes.

As for the claimed super-regularity of the vertical pairs, let $A_i$, $B_i$ and $B_{i+1}$ be clusters at an arbitrary step of the algorithm. Using Proposition 4 and (1) we know that the pairs $(A_i, B_i)$ and $(A_i, B_{i+1})$ being $(\hat{\varepsilon}, \hat{d})$-regular implies that there are at least $(1 - \hat{\varepsilon})|A_i|$ vertices in $A_i$ having at least $(\hat{d} - \hat{\varepsilon})|B_{i+1}| - t \geq (\hat{d} - \hat{\varepsilon})|B_{i+1}| - 2k\xi n$ neighbours in $B_{i+1}$, and it remains to prove that $(\hat{d} - \hat{\varepsilon})|B_{i+1}| - 2k\xi n \geq d|B_{i+1}|$ which is equivalent to $2k\xi n/|B_{i+1}| \leq 100k^2\sqrt{\xi} - 64k^2\xi - 24k\xi$. Because of $2k\xi n/|B_{i+1}| \leq 2k\xi n/(|B'_{i+1}| - t) \leq 2k\xi n/(n/2k - 2k\xi n) = 4k^2\xi/(1 - 4k^2\xi)$ it is therefore sufficient that $4k^2\xi/(1 - 4k^2\xi) \leq 100k^2\sqrt{\xi} - 64k^2\xi - 24k\sqrt{\xi}$ and it is easy to check that this is true by the hypothesis on $\xi$. $\square$

Now we will prove Lemma 9. To this end we will apply Lemma 5 to the input graph $G$. By $(a)$ and $(b)$ of Lemma 5 we obtain a regular partition with a bipartite reduced graph $R$ of high minimum degree. Theorem 2 then guarantees the existence of a Hamilton cycle in $R$ which will imply property (G3). This Hamilton cycle serves as $R^*$ in Lemma 5$(c)$ which promises a regular partition of $G$ that is super-regular on $R^*$. For finishing the proof we will use a greedy strategy for distributing the vertices into the exceptional sets over the clusters of this partition (without destroying the super-regularity required for (G2)) and then apply Lemma 11 to adjust the cluster sizes as needed for (G1).

*Proof of Lemma 9.* Let $\gamma > 0$ given. We assume without loss of generality that $\gamma < 1/20$ and set $d_{\text{LG}} := \gamma^2/100$. Now let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ be given. We assume that $\varepsilon \le \gamma^2/1000$, since otherwise we can set $\varepsilon := \gamma^2/1000$, prove the lemma, and all statements will still hold for any larger $\varepsilon$.

Our next task is to choose $\varepsilon'$ and $d'$. For this, consider the following functions in $\varepsilon'$ and $d'$:

$$\begin{aligned} \varepsilon'' &:= \frac{\varepsilon'}{1-2\varepsilon'}\,, & \hat{\varepsilon} &:= \varepsilon'' + 6\sqrt{\varepsilon''/\gamma(1-\varepsilon'')}\,, \\ d'' &:= d' - 4\varepsilon'\,, & \hat{d} &:= d'' - 4\varepsilon''/\gamma(1-\varepsilon'')\,. \end{aligned} \tag{2}$$

Observe that

$$\varepsilon' \lll \varepsilon'' \lll \hat{\varepsilon} \qquad \text{and} \qquad \hat{d} \lll d'' \lll d'\,,$$

by which we mean, for example, that $\varepsilon' \le \varepsilon''$ but that we can make $\varepsilon''$ arbitrarily small by choosing $\varepsilon'$ sufficiently small. Keeping in mind that $\gamma < 1/20$, it is easy to check that when setting $\varepsilon' := \varepsilon^3\gamma^3$ and $d' := \varepsilon + \gamma^2$, the following inequalities are all satisfied:

$$\hat{\varepsilon} \le \tfrac{1}{10}\varepsilon\,, \qquad \hat{d} - \varepsilon \ge 2d_{\text{LG}}\,, \qquad \gamma - d' - \varepsilon'' > 0 \tag{3}$$

$$(\tfrac{1}{2} + \gamma - \varepsilon'')(1-d'')^{-1} \ge \tfrac{1}{2} + \tfrac{2}{3}\gamma\,, \qquad d''(1-d'')^{-1} \le \tfrac{1}{6}\gamma\,. \tag{4}$$

Next, using (3), we can choose an integer $k_0'$ with $k_0 \le k_0'$ such that for all integers $k$ with $k_0' \le k$ we have

$$(\gamma - d' - \varepsilon'')k \ge 1\,. \tag{5}$$

Apply Lemma 5 with $\varepsilon'$, $\Delta := 2$, and with $k_0$ replaced by $k_0'$, to obtain $K_0$. Choose $\xi_{\text{LG}} > 0$ such that

$$100K_0\sqrt{\xi_{\text{LG}}} \le \tfrac{1}{10}\varepsilon, \quad 100(K_0)^2\sqrt{\xi_{\text{LG}}} \le d_{\text{LG}}. \tag{6}$$

Now let $G$ be given. Feed $d'$ and $G$ into Lemma 5 and obtain $k \in \mathbb{N}$ with $k_0 \le k_0' \le k \le K_0$ together with an equipartition of $G$ into $2k+2$ classes and an $(\varepsilon', d')$-reduced graph $R$ on $2k$ vertices by $(a)$ of Lemma 5. By assumption $\delta(G) \ge (\frac{1}{2}+\gamma)n$, so setting $\nu := 1/2 + \gamma$ and making use of part $(b)$ of Lemma 5, we get

$$\delta(R) \ge (\tfrac{1}{2} + \gamma - d' - \varepsilon'')|V(R)| = \tfrac{1}{2}|V(R)| + (\gamma - d' - \varepsilon'')k \overset{(5)}{\ge} \tfrac{1}{2}|V(R)| + 1.$$

We infer from Theorem 2 that $R$ contains a Hamilton cycle $R^*$. Now apply part $(c)$ of Lemma 5 and obtain an equipartition of $G$ which is $(\varepsilon'', d'')$-regular on $R$, $(\varepsilon'', d'')$-super-regular on $R^*$, and has classes

$$A = A_0''\dot\cup \ldots \dot\cup A_k'' \quad \text{and} \quad B = B_0''\dot\cup \ldots \dot\cup B_k''.$$

Obviously, $R$ and thus $R^*$ are bipartite and so, without loss of generality (renumbering the clusters if necessary), we can assume that the Hamilton cycle $R^*$ consists of the vertices representing the classes

$$A_1'', B_2'', A_2'', B_3'', \dots, B_k'', A_k'', B_1'', A_1''$$

with edges in this order. Therefore, we know that the pairs $(A_i'', B_i'')$ and $(A_i'', B_{i+1}'')$ are $(\varepsilon'', d'')$-super-regular for all $i \in [k]$. Let $L := |A_i''| = |B_i''|$ and observe that

$$(1-\varepsilon'')\frac{n}{k} \leq L \leq \frac{n}{k}\,.$$

Our next aim is to get rid of the classes $A_0''$ and $B_0''$ by moving their vertices to other classes. We will do this, roughly speaking, as follows. When moving a vertex $x \in A_0''$ to some class $A_i''$, say, we will move an arbitrary vertex $y \in B_0''$ to the corresponding class $B_i''$ at the same time. We will also make sure that $x$ has at least $d''|B_i''|$ neighbours in $B_i''$ and $y$ has at least $d''|A_i''|$ neighbours in $A_i''$. Here are the details for this procedure. For an arbitrary pair $(x, y) \in A_0'' \times B_0''$ we define

$$I(x,y) := \Big\{ i \in [k] \colon \quad |N_G(x) \cap B_i''| \geq d''\ |B_i''| \quad \text{and} \quad |N_G(y) \cap A_i''| \geq d''\ |A_i''| \Big\}\,.$$

We claim that for every $(a, b) \in A_0'' \times B_0''$ we have $|I(x, y)| \geq \gamma k$. To prove this claim, first recall that $L = |A_i''| = |B_i''|$ for all $i \in [k]$. Define

$$\begin{aligned} I(x) &:= \big\{ i \in [k] \colon |N_G(x) \cap B_i''| \geq d''|B_i''| \big\}\,, \\ I(y) &:= \big\{ i \in [k] \colon |N_G(y) \cap A_i''| \geq d''|A_i''| \big\}\,. \end{aligned}$$

As $|A_0''| = |B_0''| \leq \varepsilon'' n$ we have

$$\begin{aligned} (\tfrac{1}{2}+\gamma)n \ \leq\ \deg_G(x) \ &\leq\ |I(x)|L + (k - |I(x)|)\, d''L + \varepsilon'' n \\ &= |I(x)|(1-d'')L + kd''L + \varepsilon'' n\,. \end{aligned}$$

and hence

$$\begin{aligned} |I(x)| &\geq \frac{(\frac{1}{2}+\gamma)n - kd''L - \varepsilon'' n}{(1-d'')L} = \frac{(\frac{1}{2}+\gamma-\varepsilon'')}{1-d''}\frac{n}{L} - \frac{d''}{1-d''}\,k \\ &\overset{(4)}{\geq} (\tfrac{1}{2}+\tfrac{2}{3}\gamma)k - \tfrac{1}{6}\gamma k = (\tfrac{1}{2}+\tfrac{1}{2}\gamma)k\,. \end{aligned}$$

Similarly, $|I(y)| \geq (\frac{1}{2}+\frac{1}{2}\gamma)k$. Since $I(x)$ and $I(y)$ are both subsets of $[k]$, this implies that $|I(x, y)| = |I(x) \cap I(y)| \geq \gamma k$, which proves the claim.

We group the vertices in $A_0'' \cup B_0''$ into (at most $\varepsilon'' n$) pairs $(x, y) \in A_0'' \times B_0''$ and choose an index $i \in I(x, y)$ which has the property that $(A_i'', B_i'')$ has so far received a minimal number of additional vertices. Then we move $x$ into $A_i''$ and $y$ into $B_i''$. Hence, at the end, every cluster $A_i''$, or $B_i''$ gains at most $\varepsilon'' n/(\gamma k)$ additional vertices. Denote the final partition obtained in this way by

$$A \dot\cup B = \hat{A}_1 \dot\cup \hat{B}_1 \dot\cup \dots \dot\cup \hat{A}_k \dot\cup \hat{B}_k\,.$$

Set $\alpha := \beta := \varepsilon''/\gamma(1-\varepsilon'')$ and observe that

$$\frac{\varepsilon'' n}{\gamma k} = \alpha(1-\varepsilon'')\frac{n}{k} \leq \alpha L\,.$$

So Proposition 8 tells us that for all $i \in [k]$ the pairs $(\hat{A}_i, \hat{B}_i)$ are still $(\hat{\varepsilon}, \hat{d})$-super-regular and the pairs $(\hat{A}_i, \hat{B}_{i+1})$ are still $(\hat{\varepsilon}, \hat{d})$-regular, because

$$\hat{\varepsilon} \overset{(2)}{=} \varepsilon'' + 6\sqrt{\varepsilon''/\gamma(1-\varepsilon'')} = \varepsilon'' + 3(\sqrt{\alpha} + \sqrt{\beta}) \qquad \text{and}$$
$$\hat{d} \overset{(2)}{=} d'' - 4\varepsilon''/\gamma(1-\varepsilon'') = d'' - 4\alpha = d'' - 2(\alpha+\beta)\,.$$

Now we return to the statement of Lemma 9. We set $n_i := |\hat{A}_i| = |\hat{B}_i|$ for all $i \in [k]$. Let $(a_i)_{i\in[k]}$ and $(b_i)_{i\in[k]}$ be given and set $a''_i := a_i - n_i$ and $b''_i := b_i - n_i$. Then

$$a''_i \le \xi_{\mathrm{LG}} n, \quad b''_i \le \xi_{\mathrm{LG}} n, \quad \sum_{i\in[k]} a''_i = \sum_{i\in[k]} a_i - \sum_{i\in[k]} n_i = n - n = 0 = \sum_{i\in[k]} b''_i\,.$$

Therefore we can apply Lemma 11 with parameter $\xi_{\mathrm{LG}}$ to the graph $G$ with partitions $\hat{A}_1 \dot\cup \ldots \dot\cup \hat{A}_k$ and $\hat{B}_1 \dot\cup \ldots \dot\cup \hat{B}_k$. Since

$$\hat{\varepsilon} + 100k\sqrt{\xi_{\mathrm{LG}}} \overset{(3),(6)}{\le} \tfrac{1}{10}\varepsilon + \tfrac{1}{10}\varepsilon \le \varepsilon \qquad \text{and}$$
$$\hat{d} - 100k^2\sqrt{\xi_{\mathrm{LG}}} - \varepsilon \overset{(3),(6)}{\ge} 2d_{\mathrm{LG}} - d_{\mathrm{LG}} = d_{\mathrm{LG}}\,,$$

we obtain sets $A_i$ and $B_i$ for each $i \in [k]$ such that $|A_i| = |\hat{A}_i| + a''_i = n_i + a''_i = a_i$ and $|B_i| = b_i$, and with the property that $(A_i, B_i)$ is $(\varepsilon, d)$-super-regular and $(A_i, B_{i+1})$ is $(\varepsilon, d)$-regular. This completes the proof of Lemma 9. □

## 5. Distributing $H$ among the edges of a cycle

In this section we will provide the proof of the Lemma for $H$ (Lemma 10). The idea is to cut $H$ into small pieces along its bandwidth ordering, that is, an ordering of the vertices $H$ that respects the bandwidth bound. These pieces are then distributed to the edges $A_iB_i$ of the cycle $C$ in such a way that the following holds. Let $X_i$ be all the vertices from $X$, and $Y_i$ all the vertices from $Y$ that were assigned to the edge $A_iB_i$. Then we require that $X_i$ and $Y_i$ are roughly of size $n_i$. Observe that this goal would be easy to achieve if $H$ were *locally balanced*, i.e., if each of the small pieces had colour classes of equal size. While this need not be the case, we know, however, that $H$ itself is a *balanced* bipartite graph. Therefore we use a probabilistic argument to show that the pieces of $H$ can be grouped in such a way that the resulting packages form balanced bipartite subgraphs of $H$. The details of this argument are given in Section 5.1.

After this distribution of the pieces to the edges $A_iB_i$ we will construct the desired homomorphism $f$ in the following way. We will map most vertices of $X_i$ to $A_i$ and most vertices of $Y_i$ to $B_i$.

5.1. **Balancing $H$ locally.** Our goal is to group small pieces $W_1, \ldots, W_\ell$ of the balanced bipartite graph $H$ on $2n$ vertices into packages $P_1, \ldots, P_k$ that form balanced bipartite subgraphs of $H$. This is equivalent to the following problem. Given the sizes $a_j$ and $b_j$ of the colour classes of each piece $W_j$ (i.e., $a_j$ counts the vertices of $W_j$ that are in $X$ and $b_j$ those that are in $Y$) we know that the $a_j$'s sum up to $n$ and the $b_j$'s sum up to $n$. Then we would like to have a mapping $\varphi : [\ell] \to [k]$ such that for all $i \in [k]$ the $a_j$ with $j \in \varphi^{-1}(i)$ sum up approximately to the same value as the $b_j$ with $j \in \varphi^{-1}(i)$. The following lemma asserts that such a mapping $\varphi$ exists. The package $P_i$ will then (in the proof of Lemma 10) consist of all pieces $W_j$ with $j \in \varphi^{-1}(i)$.

**Lemma 12.** *For all $0 < \xi \le 1/4$ and all positive integers $k$ there exists $\ell \in \mathbb{N}$ such that for all integers $n \ge \ell$ the following holds. Let $(n_i)_{i\in[k]}$, $(a_j)_{j\in[\ell]}$, and $(b_j)_{j\in[\ell]}$ be integer partitions of $n$ such that $n_i \le \frac{1}{8}n$ and $a_j + b_j \le (1+\xi)\frac{2n}{\ell}$ for all $i \in [k]$, $j \in [\ell]$. Then there is a map $\varphi : [\ell] \to [k]$ such that for all $i \in [k]$ and $\bar{a}_i := \sum_{j\in\varphi^{-1}(i)} a_j$ and $\bar{b}_i := \sum_{j\in\varphi^{-1}(i)} b_j$ we have*

$$\bar{a}_i < n_i + \xi n \qquad \text{and} \qquad \bar{b}_i < n_i + \xi n\,. \tag{7}$$

In the proof of Lemma 12 we will use a Chernoff bound and the following formulation of a concentration bound due to Hoeffding.

**Theorem 13** (Hoeffding bound [2, Theorem A.1.16])**.** *Let $X_1, \dots, X_s$ be independent random variables with $\mathbb{E}X_i = 0$ and $|X_i| \le 1$ for all $i \in [s]$ and let $X$ be their sum. Then $\mathbb{P}[|X| \ge a] \le 2\exp(-a^2/(2s))$.* □

*Proof of Lemma 12.* For the proof of this lemma we use a probabilistic argument and show that under a suitable probability distribution a random map satisfies the desired properties with positive probability. For this purpose set $\ell := \lceil 1000k^5/\xi^2 \rceil$ and construct a random map $\varphi\colon [\ell] \to [k]$ by choosing $\varphi(j) = i$ with probability $n_i/n$ for $i \in [k]$, independently for each $j \in [\ell]$. To show that this map satisfies (7) with positive probability we first estimate the sum of all $a_j$'s and $b_j$'s assigned to a fixed $i \in [k]$. To this end, let $\mathbb{1}_j$ be the indicator variable for the event $\varphi(j) = i$ and define a random variable $S_i := \sum_{j\in[\ell]} \mathbb{1}_j$. Clearly $S_i$ is binomially distributed, we have $\mathbb{E}S_i = \ell\frac{n_i}{n}$, and by the Chernoff bound $\mathbb{P}[|S_i| \ge \mathbb{E}S_i + t] \le 2\exp(-2t^2/\ell)$ (cf. [11, Remark 2.5]) we get

$$\mathbb{P}\Big[\big|S_i - \ell\frac{n_i}{n}\big| \ge \tfrac{1}{2}\xi\ell\Big] \le 2\exp(-\tfrac{1}{2}\xi^2\ell).$$

Next, we examine the difference between the sum of the $a_j$'s assigned to $i$ and the sum of the $b_j$'s assigned to $i$. We define random variables $D_{i,j} := \frac{\ell}{3n}(a_j - b_j)(\mathbb{1}_j - \frac{n_i}{n})$ and set $D_i := \sum_{j\in[\ell]} D_{i,j}$. Then $\mathbb{E}D_{i,j} = 0$ and as $a_j + b_j \le \frac{3n}{\ell}$ we have $|D_{i,j}| \le 1$. Thus Theorem 13 implies

$$\mathbb{P}\big[|D_i| \ge \tfrac{1}{6}\xi\ell\big] \le 2\exp(-\tfrac{1}{72}\xi^2\ell).$$

By the union bound, the probability that we have

$$|S_i - \ell\tfrac{n_i}{n}| < \tfrac{1}{2}\xi\ell \qquad \text{and} \qquad |D_i| < \tfrac{1}{6}\xi\ell \qquad \text{for all } i \in [k] \tag{8}$$

is therefore at least $1 - k\cdot 2\exp(-\frac{1}{2}\xi^2\ell) - k\cdot 2\exp(-\frac{1}{72}\xi^2\ell)$ which is strictly greater than 0 by our choice of $\ell$. Therefore there exists a map $\varphi$ with (8). We claim that this map satisfies (7). To see this, observe first that $\frac{3n}{\ell}D_i = \sum_{j\in\varphi^{-1}(i)}(a_j - b_j) = \bar{a}_i - \bar{b}_i$ which together with (8) implies $\bar{a}_i - \bar{b}_i < \xi n$. Moreover, we have $S_i = |\varphi^{-1}(i)|$ and

$$\begin{aligned}\bar{a}_i &= \tfrac{1}{2}(\bar{a}_i + \bar{b}_i) + \tfrac{1}{2}(\bar{a}_i - \bar{b}_i) \le \tfrac{1}{2}(1+\xi)\tfrac{2n}{\ell}|\varphi^{-1}(i)| + \tfrac{1}{2}\cdot\tfrac{1}{2}\xi n \\ &\overset{(8)}{<} \frac{1}{2}(1+\xi)\frac{2n}{\ell}\Big(\ell\frac{n_i}{n} + \frac{1}{2}\xi\ell\Big) + \frac{1}{4}\xi n \le n_i + \xi n\end{aligned}$$

where the last inequality follows from $\xi \le \frac{1}{4}$ and $n_i \le \frac{1}{8}n$. Since an entirely analogous calculation shows that $\bar{b}_i < n_i + \xi n$, this completes the proof of (7). □

5.2. **The proof of the Lemma for $H$.** For the proof of Lemma $H$ we will now use Lemma 12 as outlined in the beginning of Section 5.1. In this way we obtain an assignment of pieces $W_1, \dots W_\ell$ of $H$ to edges $A_iB_i$ of $C$. This assignment, however, does not readily give a homomorphism from $H$ to $C$ as there might be edges between pieces $W_j$ and $W_{j+1}$ that end up on edges $A_iB_i$ and $A_{i'}B_{i'}$ which are not neighbouring in $C$. Nevertheless (owing to the small bandwidth of $H$) we will be able to transform it into a homomorphism by assigning some few vertices of $W_{j+1}$ to other vertices of $C$ along the path between $A_iB_i$ and $A_{i'}B_{i'}$ in $C$.

*Proof of Lemma 10.* Let $k$ and $\xi$ be given. Give $\xi' := \xi/4$ and $k$ to Lemma 12, get $\ell$, set $\beta := \xi'/(4\ell k)$ and $n_0 := \lceil \ell/(2\xi) \rceil$, and let $H$ and $(n_i)_{i\in[k]}$ be given as in the statement of the lemma for $H$.

We assume that the vertices of $H$ are given a bandwidth labelling, partition $V(H)$ along this labelling into $\ell$ sets $W_1, \dots, W_\ell$ of as equal sizes as possible and define $x_i := |W_i \cap X|$ and $y_i := |W_i \cap Y|$. Then $x_i + y_i = |W_i| \le \lceil 2n/\ell \rceil \le 2n/\ell + 1 \le (1+\xi)2n/\ell$ and since $n_i \le n/8$ by hypothesis we can give $(n_i)_{i\in[k]}$, $(x_i)_{i\in[\ell]}$ and $(y_i)_{i\in[\ell]}$ to Lemma 12 and get a $\varphi\colon [\ell] \to [k]$ with (7).

Let us discuss the main difficulty in our proof. Since the map $\varphi$ is obtained via the probabilistic method, there is no control over how far apart in the Hamilton cycle $C$ two sets $W_{\varphi(i-1)}$ and $W_{\varphi(i)}$ will be assigned by $\varphi$. Hence these sets might end up in non-adjacent vertices of the cycle $C$. If there are edges between $W_{\varphi(i-1)}$ and $W_{\varphi(i)}$ we need to guarantee, however, that these edges are mapped to edges of $C$ in order to obtain the desired homomorphism $f$. Therefore, we resort to a greedy linking process which robs the pieces $W_i$ of a small number of vertices. These are then distributed over the clusters lying between the cluster pair $A_{\varphi(i-1)}, B_{\varphi(i-1)}$ and the cluster pair $A_{\varphi(i)}, B_{\varphi(i)}$ such that the corresponding edges of $H$ are placed on edges of $C$.

Let $w_i$ be the first vertex in $W_i$ and define sets of *linking vertices* by

$$L^i_j := [w_i + (j-1)\beta n, w_i + j\beta n) \subseteq W_i$$

for every $j \in [2k]$ , and set $L^i := \bigcup_{j\in[2k]} L^i_j$. Then all $L^i_j$ have the common cardinality $\beta n$ and $|L^i| = 2k\beta n$. Since $\beta \le 1/(4k\ell)$ implies that $2k\beta n + \beta n \le \lfloor 2n/\ell \rfloor \le |W_i|$ for every $i \in [\ell]$, we have $L^i \subsetneq W_i$ for every $i \in [\ell]$ where $|W_i \backslash L^i| \ge \beta n$, i.e., at the end of every set $W_i$ there are at least $\beta n$ non-linking vertices (see the left hand side of Figure 2).

We now construct a map $f\colon V(H) \to \{A_1, \dots, A_k, B_1, \dots, B_k\}$ by defining, for every $i \in [\ell]$,

$$f(x) := \begin{cases} A_{\varphi(i-1)+\lfloor j/2 \rfloor} & \text{if } x \in L^i_j \text{ with } j \in \big[2\cdot\big((\varphi(i)-\varphi(i-1)) \mod k\big)\big], \\ A_{\varphi(i)} & \text{else,} \end{cases} \tag{9}$$

for every $x \in W_i \cap X$, and

$$f(y) := \begin{cases} B_{\varphi(i-1)+\lceil j/2 \rceil} & \text{if } y \in L^i_j \text{ with } j \in \big[2\cdot\big((\varphi(i)-\varphi(i-1)) \mod k\big)\big], \\ B_{\varphi(i)} & \text{else,} \end{cases} \tag{10}$$

for every $y \in W_i \cap Y$, and show that this is indeed a homomorphism (see also Figure 2). To do this, it is convenient to note that a set $\{A_i, B_{i'}\}$ is an edge of $C$ if and only if $0 \le i' - i \le 1$.

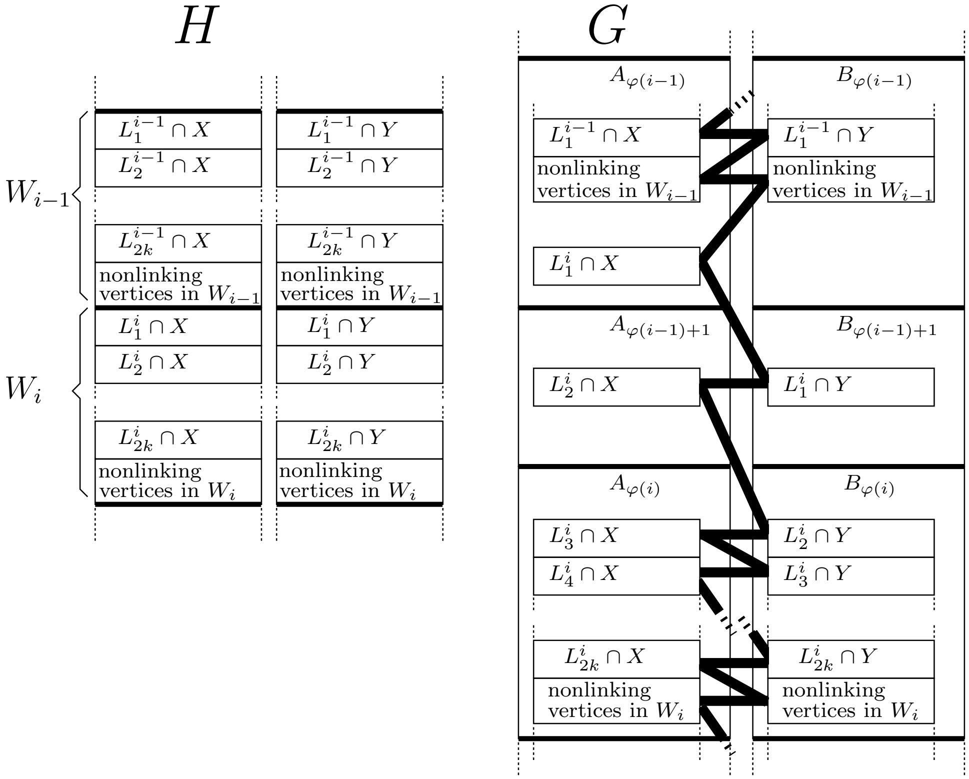

Figure 2. The linking procedure.

Let arbitrary vertices $x \in X$ and $y \in Y$ with $\{x, y\} \in F$ be given. Since the sets $W_i$ are defined along the bandwidth labelling, either $x$ and $y$ are both within the same $W_i$, or $x$ and $y$ lie in consecutive sets $W_i$ and $W_{i+1}$. We will now distinguish several cases. For brevity let $I_i := \big[2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)\big]$.

*Case 1.* Both $x$ and $y$ lie within the same set $W_i$.
*Case 1.1.* There is $j \in I_i$ with $x \in L^i_j$, hence $f(x) = A_{\varphi(i-1)+\lfloor j/2 \rfloor}$. Due to the bandwidth condition together with $|L^i_j| = \beta n$, if $y \notin L^i_j$ and $j+1 \in I_i$, then necessarily $y \in L^i_{j+1}$, which explains the following three sub-cases.
*Case 1.1.1.* We have $y \in L^i_j$, hence $f(y) = B_{\varphi(i-1)+\lceil j/2 \rceil}$, hence the difference of the indices of $f(x)$ and $f(y)$ is $\lceil j/2 \rceil - \lfloor j/2 \rfloor$, which is either 0 or 1 according to whether $j$ is even or odd, hence $\{f(x), f(y)\} \in E(C)$.
*Case 1.1.2.* We have $y \notin L^i_j$ and $j+1 \in I_i$, hence $y \in L^i_{j+1}$, hence $f(y) = \varphi(i-1) + \lceil (j+1)/2 \rceil$, hence the difference of indices of $f(y)$ and $f(x)$ is $\lceil (j+1)/2 \rceil - \lfloor j/2 \rfloor$, and this is always 1, whether $j$ is even or odd, so $\{f(x), f(y)\} \in E(C)$.
*Case 1.1.3.* We have $y \notin L^i_j$ and $j+1 \notin I_i$, hence $f(y) = B_{\varphi(i)}$. Here, $j+1 \notin L^i_j$ implies that $j \geq 2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)$ while being within Case 1.1 implies $j \in I_i$, hence $j \leq 2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)$, so we have $j = 2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)$, thus $f(x) = A_{\varphi(i-1)+\lfloor j/2 \rfloor} = A_{\varphi(i)}$, the index difference between $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$.
*Case 1.2.* There is no $j \in I_i$ with $x \in L^i_j$, hence $f(x) = A_{\varphi(i)}$. Being within Case

1, i.e. $y \in W_i$, it follows that there are exactly two cases.
*Case 1.2.1.* If $y$ precedes $x$ in the bandwidth labelling, then $y \in L^i_{2\cdot q}$ with $q = (\varphi(i) - \varphi(i-1)) \mod k$. Hence $f(y) = B_{\varphi(i)}$, so the index difference between $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$.
*Case 1.2.2.* If $y$ succeeds $x$ in the bandwidth labelling, then, since $y \in W_i$ by being within Case 1, there is no $j \in I_i$ with $y \in I_i$, hence $f(y) = B_{\varphi(i)}$, so again the index difference between $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$.

*Case 2.* We have $x \in W_i$ and $y \in W_{i+1}$. Then, by the bandwidth condition and size of the sets of linking vertices, we must have $y \in L_1^{i+1}$, hence $f(y) = B_{\varphi((i+1)-1)+\lceil 1/2 \rceil} = B_{\varphi(i)+1}$, and since there are at least $\beta n$ non-linking vertices to the right of $W_i$, the vertex $x$ cannot lie in a $L^i_j$, hence $f(x) = A_{\varphi(i)}$, so the index difference of $f(y)$ and $f(x)$ is 1 and $\{f(x), f(y)\} \in E(C)$.

*Case 3.* We have $y \in W_i$ and $x \in W_{i+1}$. Then, by the bandwidth condition and size of the sets of linking vertices, we must have $x \in L_1^{i+1}$, hence $f(x) = A_{\varphi((i+1)-1)+\lfloor 1/2 \rfloor} = A_{\varphi(i)}$, and since there are at least $\beta n$ non-linking vertices to the right of $W_i$, the vertex $y$ cannot lie in a $L^i_j$, hence $f(y) = B_{\varphi(i)}$, so the index difference of $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$. This completes the proof that $f$ is a homomorphism.

We now prove (H1) and (H2). Define $S := \bigcup_{i\in[\ell]} L^i$. Then $|S| \le \ell \cdot 2k \cdot \beta n \le \ell \cdot 2k \cdot (\xi'/(2\ell k)) \cdot n = \xi' n \le \xi n$, which shows (H1), and (H2) is obvious from the definitions of $S$ and the map $f$ above.

We now prove (H3). For this it suffices to note, rather crudely, that for every $j \in [k]$, no pre-image $f^{-1}(A_j)$ can become larger than the sum of the sizes of all sets $W_i$ assigned to $A_j$ by $\varphi$ (which by the definition of $f$ equals the sum of all $x_i = |X \cap W_i|$ with $\varphi(i) = j$) plus the total number of linking vertices, i.e. for every $j \in [k]$, using the choice of $\beta$ and using that $\varphi$ has the property promised by Lemma 12, we have $|f^{-1}(A_j)| \le \left(\sum_{i\in\varphi^{-1}(j)} x_i\right) + |\bigcup_{i\in[\ell]} L^i| \le n_j + \xi' n + \ell \cdot |L^i| = n_j + \xi' n + 2k\ell\beta n \le n_j + 2\xi' n = n_j + \xi n$, completing the proof of (H3). □

## 6. Concluding remarks

**Unbalanced $H$ and $G$.** Essentially the same proof allows for an analogue of Theorem 3 for bipartite graphs $H$ and $G$ that are not balanced but whose colour classes have the same sizes. More precisely, let $H = (X\dot\cup Y, F)$ and $G = (A\dot\cup B, E)$ be as in Theorem 3, except that $|X| = |A| = n_1$ and $|Y| = |B| = n_2$ (where $n_1 + n_2 = 2n$) and the minimum degree condition on $G$ is replaced by the following condition. For all $v \in A$ we have $\deg_G(v) \ge (\frac{1}{2} + \gamma)n_2$ and for all $w \in B$ we have $\deg_G(w) \ge (\frac{1}{2} + \gamma)n_1$. Then $H$ is a subgraph of $G$.

**Generating systems for the cycle space.** As an application of Theorem 3, one can show the following result. For every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ every balanced bipartite graph $G$ on $2n$ vertices with $\delta(G) \ge (\frac{1}{2} + \gamma)n$ has the property that the edge-sets of all Hamilton cycles in $G$ form a generating system for the cycle space of $G$. A proof for this theorem will be given in a forthcoming paper [9]. It utilises the fact that a special balanced bipartite graph $H$ (the so-called Möbius ladder) of bounded maximum degree and bandwidth has this property and then shows that this gets translated to the graph $G$, using a result of Locke [16].

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