On Popular Random Assignments

Haris Aziz, Felix Brandt, and Paul Stursberg

[1] NICTA and University of New South Wales, Sydney, NSW 2033, Australia
haris.aziz@nicta.com.au
[2] Institut für Informatik, Technische Universität München, 85748 München, Germany
brandtf@in.tum.de
[3] Zentrum Mathematik, Technische Universität München, 85747 München, Germany
stursb@ma.tum.de

Abstract. One of the most fundamental and ubiquitous problems in microeconomics and operations research is how to assign objects to agents based on their individual preferences. An assignment is called popular if there is no other assignment that is preferred by a majority of the agents. Popular assignments need not exist, but the minimax theorem implies the existence of a popular *random* assignment. In this paper, we study the compatibility of popularity with other properties that have been considered in the literature on random assignments, namely efficiency, equal treatment of equals, envy-freeness, and strategyproofness.

1 Introduction

One of the most fundamental and ubiquitous problems in microeconomics and operations research is how to assign objects to agents based on their individual preferences (see, e.g., [21, 4, 5]). In its simplest form, the problem is known as the *assignment problem*, the *house allocation problem*, or *two-sided matching with one-sided preferences*. Formally, the *assignment problem* concerns a set of agents $A = \{a_1, \ldots, a_n\}$ and a set of houses $H = \{h_1, \ldots h_n\}$. Each agent has preferences over the elements of H and the goal is to assign or allocate exactly one house to each agent in an efficient and fair manner. An important assumption in this setting is that monetary transfers between the agents are not permitted.[4] The assignment problem has numerous applications in a variety of settings such the assignment of dormitories to students, jobs to applicants, rooms to housemates, processor time slots to jobs, parking spaces to employees, offices to workers, kidneys to patients, school seats to student applicants, etc. Clearly, deterministic assignments may fail to satisfy even extremely mild fairness criteria such as equal treatment of equals. It is therefore an established practice to restore (*ex ante*) fairness by introducing randomization. Random assignments

[4] Monetary transfers may be impossible or highly undesirable, as is the case if houses are public facilities provided to low-income people. There are a number of settings such as voting, kidney-exchange, or school choice in which money cannot be used as compensation due to practical, ethical, or legal constraints (see, e.g., [20]).

This paper was published as H. Aziz, F. Brandt, and P. Stursberg. On popular random assignments. In B. Vöcking, editor, *Algorithmic Game Theory*, volume 8146 of *Lecture Notes in Computer Science*, pages 183-194. Springer Berlin Heidelberg, 2013. The final publication is available at
http://link.springer.com/chapter/10.1007/978-3-642-41392-6_16

are strongly related to fractional assignments and random assignment rules can also be used to fractionally allocate resources to agents.

A deterministic assignment (or matching) is deemed *popular* if there exists no other assignment that a majority of agents prefers to the former (see, e.g., [1, 3, 17, 12]). Popular assignments were first considered by Gärdenfors [10]. While popular assignments can be computed in polynomial time [1], they unfortunately may not exist. Taking cue from this observation, McCutchen [17] proposed two quantities—the *unpopularity margin* and the *unpopularity factor*—to measure the unpopularity of an assignment and defined the notion of a *least unpopular* assignment, which is guaranteed to exist.[5] However, computing least unpopular assignments turned out to be NP-hard. Alternatively, Kavitha et al. [13] suggested the notion of popular *random* assignments. A random assignment p is popular if there is no other assignment q such that the expected number of agents who prefer the outcome of q to that of p is greater than $n/2$. Kavitha et al. [13] showed that popular random assignments not only exist due to the minimax theorem but can also be computed in polynomial time via linear programming. To the best of our knowledge, axiomatic properties of popular random assignments have not been studied so far. In this paper, we aim at improving our understanding of popular random assignments by investigating which common axiomatic properties are compatible with popularity.

Contributions. We first point out that popular random assignments can be viewed as a special case of *maximal lotteries*, which were proposed in the context of social choice by Fishburn [8].[6] Assignment can be seen as a restricted domain of social choice in which each alternative corresponds to an assignment. Preferences over houses can be easily extended to preferences over assignments by assuming that each agent only cares about the house assigned to himself and is indifferent between all assignments in which he is assigned the same house. We prove the following statements.

- Every popular assignment is efficient.
- There always exists a popular assignment that satisfies equal treatment of equals. Such an assignment can furthermore be computed in polynomial time.
- Popularity and envy-freeness are incompatible if $n \geq 3$. If a popular and envy-free assignment exists, it can be computed in polynomial time.
- There are no strategyproof popular random assignment rules if $n \geq 3$.

[5] The unpopularity margin of a matching is the maximum majority difference by which it is dominated by any other matching. The unpopularity factor of a matching is the maximum factor by which it is dominated by any other matching.

[6] Maximal lotteries were first considered by Kreweras [14] and independently rediscovered and studied in detail by Fishburn [8]. Interestingly, maximal lotteries or variants thereof have been rediscovered again by economists, mathematicians, political scientists, and computer scientists [15, 7, 9, 19]. Strategyproofness and efficiency of maximal lotteries were recently analyzed by Aziz et al. [2].

Related work. Random assignment rules have received enormous attention in recent years. Most notable among these rules are *random serial dictatorship (RSD)* (see e.g., [4, 16]) and the *probabilistic serial rule (PS)* [4]. Each of these rules has its own merits. However, it can be easily shown that the assignment returned by any of these rules may not be popular.

Perhaps closest to our work are the papers by Kavitha et al. [13], who introduced popular random assignments, and Bogomolnaia and Moulin [4], who outlined a systematic way of studying the properties of random assignments and random assignment rules. In particular, Bogomolnaia and Moulin [4] popularized the use of first-order stochastic dominance to formalize various notions of envy-freeness, efficiency, and strategyproofness that we also consider in this paper.

2 Preliminaries

An assignment problem is a triple $(A, H, \succsim)$ such that $A = \{a_1, \ldots, a_n\}$ is a set of agents, $H = \{h_1, \ldots, h_n\}$ is a set of houses, and $\succsim = (\succsim_1, \ldots, \succsim_n)$ is a preference profile in which $\succsim_i$ denotes an antisymmetric, complete, and transitive relation on H representing the preferences of agent i over the houses in H.[7]

A *deterministic assignment* (or *pure matching*) $M \subset A \times H = \mathscr{M}$ is a subset of non-adjacent arcs in the bipartite graph $G = (A \dot\cup H, A \times H)$. If $(i, h) \in M$, we write $M(i) = h$. A matrix $p = (p_{ih})_{(i,h)\in A\times H}$ with $p_{ih} \geq 0$, $\sum_{i\in A} p_{ih} = 1$ for all $h \in H$ and $\sum_{h\in H} p_{ih} = 1$ for all $a_i \in A$, $h \in H$ is called a *random assignment* (or *mixed matching*). Note that the entries $p_i = (p_{i1}, \ldots, p_{in})$ corresponding to arcs incident with some agent i constitute a random allocation for this agent. Further note that every random assignment may be represented by a (not necessarily unique) lottery over deterministic assignments and that in turn, every lottery over deterministic assignments induces a unique random assignment. This is known as the Birkhoff-Von Neumann theorem (see, e.g., [13]).

A natural way to compare random assignments is by means of *stochastic dominance (SD)*. Given two random assignments p and q, $p_i \succsim_i^{SD} q_i$ i.e., agent i *SD-prefers* p_i to q_i iff

$$\sum_{\substack{h\in H\\ h\succsim_i h^*}} p_{ih} \geq \sum_{\substack{h\in H\\ h\succsim_i h^*}} q_{ih} \text{ for all } h^* \in H.$$

This preference extension is of particular importance because one random assignment is SD-preferred to another iff, for any utility representation consistent with the ordinal preferences, the former yields at least as much *expected utility* as the latter (see, e.g., [11, 6]). Since for all $i \in A$, agent i compares assignment p with assignment q only with respect to his allocations p_i and q_i, we will sometimes abuse the notation by writing $p \succsim_i^{SD} q$ instead of $p_i \succsim_i^{SD} q_i$.

[7] Although we assume strict preferences for the ease of exposition, all our positive results hold for arbitrary preferences and our negative results even hold for strict preferences.

Finally, a *random assignment rule* f is a function which for each input $(A, H, \succsim)$ returns a random assignment p. When A and H are clear from the context, we simply write $f(\succsim)$ for $f(A, H, \succsim)$.

3 Desirable Properties of Random Assignment Rules

In this section, we define a number of desirable properties for random assignments and random assignment rules. Properties of assignments naturally translate to properties of assignment rules: We say that a random assignment rule f satisfies property P if every assignment p returned by f satisfies P.

Popularity. In order to define popularity, we first associate a function ϕ_i with each preference relation $\succsim_i$ on H by letting $\phi_i : H \times H \to \{-1, 0, 1\}$ such that for all $h, h' \in H$,

$$\phi_i(h, h') = \begin{cases} +1 & \text{if } h \succ_i h', \\ -1 & \text{if } h' \succ_i h, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the natural extension of ϕ_i to random assignments and take the sum over all agents. To this end, we define

$$\phi(p, q) := \sum_{a_i \in A} \sum_{h, h' \in H} p_{ih} q_{ih'} \, \phi_i(h, h')$$

and say that p is *more popular* than q if $\phi(p, q) > 0$. A random assignment p is *popular* if there is no assignment q more popular than p. It can be easily shown that both PS and RSD fail to satisfy popularity.

Efficiency. A deterministic assignment M is *Pareto efficient* if there exists no other deterministic assignment M' such that $M'(a_i) \succsim_i M(a_i)$ for all $a_i \in A$, and there exists an agent $a_i \in A$ such that $M'(a_i) \succ_i M(a_i)$. A random assignment is *ex post efficient* if it can be represented as a probability distribution over Pareto efficient deterministic assignments. Finally, a random assignment p is *SD-efficient* if there exists no assignment q such that q stochastically dominates p, i.e. $q_i \succsim_i^{SD} p_i$ for all $a_i \in A$ and $q_i \succ_i^{SD} p_i$ for some $a_i \in A$. It can be shown that SD-efficiency implies *ex post* efficiency. Furthermore, while PS satisfies SD-efficiency, RSD is only *ex post* efficient [4].

Fairness. A random assignment p satisfies *equal treatment of equals* if agents with identical preferences receive identical random allocations, i.e., $\succsim_i = \succsim_j$ implies that $p_i = p_j$ for any pair of agents i and j. Equal treatment of equals is considered as one of the most fundamental requirements in resource allocation and a "minimal test for fairness" [18]. A random assignment satisfies *SD-envy-freeness* if each agent (weakly) SD-prefers his allocation to that of any other agent. A random assignment satisfies *weak SD-envy-freeness* if no agent strictly SD-prefers someone elses allocation to his. SD-envy-freeness implies equal treatment of equals while weak SD-envy-freeness does not. PS is known to satisfy SD-envy-freeness whereas RSD only satisfies weak SD-envy-freeness [4].

Strategyproofness. In contrast to the previous conditions, strategyproofness can only meaningfully be defined as the property of an assignment *rule* rather than that of an assignment. A random assignment rule f is *SD-strategyproof* if for every preference profile $\succsim$, and for all $a_i \in A$ and $\succsim_i'$, $f(\succsim_i, \succsim_{-i}) \succsim_i^{SD} f(\succsim_i', \succsim_{-i})$. A random assignment rule f is *weakly SD-strategyproof* if for every preference profile $\succsim$, there exists no $\succsim_i'$ for some agent $a_i \in A$ such that $f(\succsim_i', \succsim_{-i}) \succ_i^{SD} f(\succsim_i, \succsim_{-i})$. RSD is SD-strategyproof whereas PS is only weakly SD-strategyproof. (When also allowing ties in the preferences, RSD remains SD-strategyproof whereas PS fails to be even weakly SD-strategyproof.)

In the remainder of this paper, we investigate whether and to which extent popularity is compatible with efficiency, fairness, and strategyproofness.

4 Efficiency

It is easy to see that popular assignments are *ex post* efficient. For the sake of contradiction let us assume that there is a deterministic assignment which is in the support of a lottery representation of some popular random assignment but which is not Pareto optimal. This implies that the deterministic assignment is Pareto dominated by another deterministic assignment and hence cannot be in the support of the popular random assignment (as replacing it by the assignment that dominates it would yield a more popular assignment).

We address SD-efficiency by first observing that popular random assignments are a special case of maximal lotteries in general social choice [8]. A lottery p is a *maximal lottery* if there exists no other lottery q for which the expected number of agents who prefer q over p is more than the expected number of agents who prefer p over q.

An assignment problem $(A, H, \succsim)$ may also be seen as a social choice problem where A is the set of agents and the alternatives to choose from are all the different (deterministic) assignments between agents in A and houses in H. The preferences of the agents over these alternatives can naturally be defined according to their preferences over the houses allocated to them (which means that agents will be indifferent between assignments that assign the same house to them). As Kavitha et al. [13] note, popularity of a random assignment p may also be defined in terms of its representation as a lottery over deterministic assignments. Furthermore, for every possible such representation the "unpopularity margin" is *equal to* that of the original assignment p. This means that every maximal lottery induces a popular random assignment, and every lottery that represents a popular assignment is maximal.

We now show that popular assignments are not only *ex post* efficient but even SD-efficient.

Lemma 1. *Let $L_p = [p^1 : M_1, \dots, p^k : M_k]$ and $L_q = [q^1 : N_1, \dots, q^k : N_l]$ be lotteries over deterministic assignments that induce the fractional assignments p and q. Then, $p \succsim_i^{SD} q$ iff $L_p \succsim_i^{SD} L_q$.*

Proof. For reasons of notational convenience, we write

$$p(M) = \begin{cases} p^j & M = M_j \in \text{supp}(L_p) \\ 0 & M \notin \text{supp}(L_p). \end{cases}$$

For every agent i and $h^* \in H$ we can pick some assignment N with $N(i) = h$ to obtain

$$\sum_{\substack{h \in H \\ h \succsim_i h^*}} p_{ih} = \sum_{\substack{M \in \mathscr{M} \\ M(i) \succsim_i h^*}} p(M) = \sum_{\substack{M \in \mathscr{M} \\ M \succsim_i N}} p(M).$$

Analogously, for every agent i and assignment N, we have

$$\sum_{\substack{M \in \mathscr{M} \\ M \succsim_i N}} p(M) = \sum_{\substack{M \in \mathscr{M} \\ M(i) \succsim_i N(i)}} p(M) = \sum_{\substack{h \in H \\ h \succsim_i N(i)}} p_{ih}.$$

This means that

$$\forall h \in H\colon \sum_{\substack{h \in H \\ h \succsim_i h^*}} p_{ih} \geq \sum_{\substack{h \in H \\ h \succsim_i h^*}} q_{ih} \quad \text{iff} \quad \forall N \in \mathscr{M}\colon \sum_{\substack{M \in \mathscr{M} \\ M \succsim_i N}} p(M) \geq \sum_{\substack{M \in \mathscr{M} \\ M \succsim_i N}} q(M),$$

i.e., $p \succsim_i^{SD} q$ iff $L_p \succsim_i^{SD} L_q$. □

Theorem 1. *Every popular assignment is SD-efficient.*

Proof. Let p be a popular assignment. Suppose that p is SD-dominated by some assignment q. Let L_p be a lottery representation of p and L_q a lottery representation of q. Then Lemma 1 implies that L_q SD-dominates L_p. But, as argued above, L_p is a maximal lottery which is a contradiction to the fact that maximal lotteries satisfy SD-efficiency (see [2]).

□

5 Equal treatment of equals

Even though popular assignments satisfy fairness in the sense of respecting majorities of agents, they can be highly unfair on the individual level. In fact, popular assignments may not even satisfy equal treatment of equals. This can be seen by considering the extremely simple case of two agents with identical preferences in which *every* random assignment is popular.

We will now show that a popular assignment that satisfies equal treatment of equals always exists and that it can be computed in polynomial time. To this end, we introduce the notion of an S-leveling:

Definition 1. *Let x be a random assignment for (A, H) and $S \subset A$. The* S-leveling *of x is the random assignment y given by*

$$y_{ah} = \begin{cases} x_{ah} & a \notin S \\ \frac{1}{|S|} \sum_{a \in S} x_{ah} & a \in S. \end{cases}$$

It is easy to see that the S-leveling of a random assignment is again a random assignment, as the sum over all edges incident to any house or agent remains unchanged.

Lemma 2. *Let x and z be random assignments for (A, H) and $S \subset A$ such that all $a \in S$ have identical preferences. Let furthermore y be the S-leveling of x. Now, the S-leveling z' of z satisfies*

$$\phi(x, z') = \phi(y, z).$$

Proof. We begin by showing that $\sum_{a \in S} (x_{ah} z'_{ah'} - y_{ah} z_{ah'}) = 0$ for all $h, h' \in H$. Let $h, h' \in H$. Then

$$\begin{aligned}
\sum_{a \in S} (x_{ah} z'_{ah'} - y_{ah} z_{ah'}) &= \sum_{a \in S} x_{ah} z'_{ah'} - \sum_{a \in S} y_{ah} z_{ah'} && (1) \\
&= \sum_{a \in S} x_{ah} \left(\frac{1}{|S|} \sum_{a \in S} z_{ah'} \right) - \sum_{a \in S} \left(\frac{1}{|S|} \sum_{a \in S} x_{ah} \right) z_{ah'} && (2) \\
&= \frac{1}{|S|} \sum_{a \in S} z_{ah'} \sum_{a \in S} x_{ah} - \frac{1}{|S|} \sum_{a \in S} x_{ah} \sum_{a \in S} z_{ah'} && (3) \\
&= 0, && (4)
\end{aligned}$$

where we use the definition of S-leveling in (2) and the fact that one of the factors in each sum does not depend on a in (3). We use the definition of function ϕ as defined in Section 3.

Now, we define $\phi^* := \phi_a$ for an arbitrary agent $a \in S$ and have $\phi^* = \phi_a$ for all agents in S due to their identical preferences. Using this notation, we show, that $\phi(x, z') - \phi(y, z) = 0$:

$$\begin{aligned}
\phi(x, z') - \phi(y, z) &= \sum_{a \in A} \sum_{h, h' \in H} x_{ah} z'_{ah'} \, \phi_a(h, h') - \sum_{a \in A} \sum_{h, h' \in H} y_{ah} z_{ah'} \, \phi_a(h, h') && (5) \\
&= \sum_{a \in A} \sum_{h, h' \in H} \phi_a(h, h') \, (x_{ah} z'_{ah'} - y_{ah} z_{ah'}) && (6) \\
&= \sum_{a \in S} \sum_{h, h' \in H} \phi_a(h, h') \, (x_{ah} z'_{ah'} - y_{ah} z_{ah'}) && (7) \\
&= \sum_{h, h' \in H} \phi^*(h, h') \sum_{a \in S} (x_{ah} z'_{ah'} - y_{ah} z_{ah'}) && (8) \\
&= 0 && (9)
\end{aligned}$$

using the fact that x and y as well as z and z' coincide on $A \setminus S$ in equation (7), the identical preferences of agents in S in (8) and finally our first claim in (9). □

Theorem 2. *There always exists a popular random assignment that satisfies equal treatment of equals. Such an assignment can furthermore be computed in polynomial time.*

Proof. Let x be a popular random assignment (the existence of which is guaranteed due to the minimax theorem) that does not satisfy equal treatment of equals for a subset A' of A and $S \subset A'$ a set of agents with identical preferences. Denote by y the S-leveling of x, which obviously has the property of treating these agents with identical preferences equally.

Suppose for contradiction that there is a random assignment z more popular than y, that is $\phi(z, y) > 0$. Using Lemma 2, we obtain a random assignment z' with $\phi(z', x) > 0$. Hence, z' is more popular than x which yields a contradiction to our assumption that x was popular.

We thus obtain a random assignment (y) that does not satisfy equal treatment of equals for a strictly smaller subset $A' \setminus S$ of A. Applying this argument iteratively, we finally obtain a random assignment that satisfies equal treatment of equals. □

To efficiently compute a popular assignment that satisfies equal treatment of equals, consider LP3 by Kavitha et al. [13] which computes a popular random assignment. With at most $O(n^2)$ extra constraints, it can be ensured that agents with same preferences get the same allocations. For each a_i, a_j such that $\succsim_i = \succsim_j$, we can impose the condition that $x(a_i, h_k) = x(a_j, h_k)$ for all $h_k \in H$. This ensures the equal treatment to equals condition.

6 Envy-freeness

In this section, we investigate to which extent popularity is compatible with envy-freeness. There are popular assignments that fail to satisfy even weak SD-envy-freeness (again, consider the case with two agents who have identical preferences). The question that we are interested in is whether, for every preference profile, there exists at least one popular assignment that satisfies SD-envy-freeness or weak SD-envy-freeness.

Theorem 3. *There exists an instance of a random assignment problem with $n = 3$ for which no popular assignment satisfies SD-envy-freeness.*

Proof. Consider the following assignment problem with three agents and three houses.

$$a_1 : h_1, h_2, h_3$$
$$a_2 : h_1, h_2, h_3$$
$$b\ : h_2, h_1, h_3$$

As noted in Section 3, any assignment that satisfies SD-envy-freeness must also satisfy equal treatment of equals. We now show that the unique popular assignment that satisfies equal treatment of equals is as follows:

$$p_{a_1h_1} = 1/2, \qquad p_{a_1h_2} = 0, \qquad p_{a_1h_3} = 1/2,$$
$$p_{a_2h_1} = 1/2, \qquad p_{a_2h_2} = 0, \qquad p_{a_2h_3} = 1/2,$$
$$p_{bh_1} = 0, \qquad p_{bh_2} = 1, \qquad p_{bh_3} = 0.$$

Consider an assignment p which satisfies equal treatment of equals. Denote by $p_1 := p_{a_1h_1} = p_{a_2h_1}$ and $p_2 := p_{a_1h_2} = p_{a_2h_2}$. Note that, in particular, as p is popular it has to be at least as popular as the pure assignment $M_1 = \{(b,h_2),(a_1,h_1),(a_2,h_3)\}$. Hence, p must fulfil $\phi(M_1,p) = 1 + p_2 - 2p_1 \leq 0$ which means that $p_1 \geq 1/2 + p_2/2$. Secondly, $1 \geq p_{a_1h_1} + p_{a_2h_1} = 2p_1$ which means that $p_1 \leq 1/2$.

The only assignment that satisfies the constraints $p_1 \geq 1/2 + p_2/2$, $p_1 \leq 1/2$, $p_1 \geq 0$, and $p_2 \geq 0$ is the one for which $p_1 = 1/2$ and $p_2 = 0$. In this assignment p, the allocations of a_1, a_2 do not SD-dominate the allocation of b according to the preference of a_1 and a_2. Therefore the only popular assignment satisfying equal treatment of equals does not satisfy SD-envy-freeness. □

Despite this negative result, an SD-envy-free popular random assignment can be computed in polynomial time whenever it exists. For each pair of agents a, b, we need the constraint that $p_a \succsim_a^{SD} p_b$. This can be encoded easily by considering at most as many partial sums as the number of houses n.

$$\sum_{\substack{h \in H \\ h \succsim_a h^*}} p_{ah} \geq \sum_{\substack{h \in H \\ h \succsim_a h^*}} p_{bh} \text{ for all } h^* \in H.$$

There are $O(n^2)$ such constraints.

Regarding *weak* SD-envy-freeness, the alternative characterization of the SD relation in terms of utility functions mentioned in Section 2 might help. This characterization allows us to ensure weak SD-envy-freeness by adding constraints to the linear program used to compute popular assignments as follows: An assignment p is *not* strictly preferred to an assignment q by agent i, if there exists some utility function u for which the expected utility of q is greater than that of p. This can be expressed by adding variables to represent the utility function u (for each agent). However, we have shown that the resulting feasible region is non-convex, which implies that this representation hardly leads to an efficient algorithm to compute such an assignment. This assessment does of course not preclude the *existence* of such an assignment.

7 Strategyproofness

Finally, we examine how popular assignment rules fare in terms of strategyproofness. It turns out that popularity is incompatible with SD-strategyproofness.

Theorem 4. *For $n \geq 3$, there are no SD-strategyproof popular randomized assignment rules.*

Proof. Consider an assignment problem with three agents and three houses and the following preferences.

$$a_1 : h_1, h_3, h_2$$
$$a_2 : h_1, h_2, h_3$$
$$a_3 : h_1, h_2, h_3$$

We show that there exists some utility function for agent a_1, compatible with his preferences, which allows him to obtain a higher expected utility if he misreports his preferences. In light of the equivalence mentioned in Section 2, this means that agent a_1 does not SD-prefer his original outcome to that which she may achieve by misreporting.

The set of all deterministic assignments is as follows:

$$\begin{aligned}
M_{123} &= \{\{a_1, h_1\}, \{a_2, h_2\}, \{a_3, h_3\}\}, \quad & M_{312} &= \{\{a_1, h_3\}, \{a_2, h_1\}, \{a_3, h_2\}\},\\
M_{231} &= \{\{a_1, h_2\}, \{a_2, h_3\}, \{a_3, h_1\}\}, \quad & M_{132} &= \{\{a_1, h_1\}, \{a_2, h_3\}, \{a_3, h_2\}\},\\
M_{321} &= \{\{a_1, h_3\}, \{a_2, h_2\}, \{a_3, h_1\}\}, \quad & M_{213} &= \{\{a_1, h_2\}, \{a_2, h_1\}, \{a_3, h_3\}\}.
\end{aligned}$$

Then consider the matrix corresponding to the pairwise weighted majority comparisons. An entry in the matrix denotes the number of agents who prefer the row assignment to the column assignment minus number of agents who prefer the column assignment to the row assignment. An assignment is popular if and only if it is a maximin strategy of the symmetric two-player zero-sum game represented by the matrix. It can be checked using an LP solver that each maximin strategy only randomizes over M_{312} and M_{321}.

Since a_1 gets h_3 in both M_{312} and M_{321}, a_1 gets h_3 with probability one in every popular assignment.

	M_{123}	M_{312}	M_{231}	M_{132}	M_{321}	M_{213}
M_{123}	0	−1	+1	0	0	0
M_{312}	+1	0	+1	0	0	+2
M_{231}	−1	−1	0	+2	−2	0
M_{132}	0	0	−2	0	−1	+1
M_{321}	0	0	+2	+1	0	+1
M_{213}	0	−2	0	−1	−1	0

Now if a_1 misreports his preferences as h_1, h_2, h_3, the new preference profile is as follows.

$$\begin{aligned}
a_1 &: h_1, h_2, h_3\\
a_2 &: h_1, h_2, h_3\\
a_3 &: h_1, h_2, h_3
\end{aligned}$$

Then, the pairwise majority margins are shown in the matrix below.

	M_{123}	M_{312}	M_{231}	M_{132}	M_{321}	M_{213}
M_{123}	0	−1	+1	0	0	0
M_{312}	+1	0	−1	0	0	0
M_{231}	−1	+1	0	0	0	0
M_{132}	0	0	0	0	−1	+1
M_{321}	0	0	0	+1	0	−1
M_{213}	0	0	0	−1	+1	0

It can be shown that a maximin strategy is a probability distribution over the following two strategies $[M_{123} : 1/3; M_{312} : 1/3; M_{231} : 1/3]$ and $[M_{132} : 1/3; M_{321} : 1/3; M_{213} : 1/3]$. Hence the induced popular assignment for any lottery corresponding to a maximin strategy is one which specifies a probability of 1/3 of each agent getting each object. Thus a_1 gets h_1, h_2, and h_3 each with probability 1/3. Now, let us assume that $u_{a_1}(h_1) + u_{a_1}(h_2) > 2u_{a_1}(h_3)$. Then a_1 gets utility $(u_{a_1}(h_1) + u_{a_1}(h_2) + u_{a_1}(h_3))/3 > 3u_{a_1}(h_3)/3 = u_{a_1}(h_3)$. □

An important open question is whether there are *weakly* SD-strategyproof popular random assignment rules. Related questions have recently also been analyzed in the more general context of social choice where it was shown that popularity is incompatible with weak SD-strategyproofness, but compatible with a significantly weaker version of weak SD-strategyproofness called *weak ST-strategyproofness* [2].

8 Conclusion

Kavitha et al. [13] have recently shown that every assignment problem admits a popular random assignment which can furthermore be computed in polynomial time using linear programming. In this paper, we investigated which common axiomatic properties are compatible with popularity. Results were mixed. It turned out that a particularly desirable aspect of popularity is that many conditions can be formalized as linear constraints that can be simply plugged into the linear program for computing popular random assignments. Furthermore, all properties considered in this paper (including popularity) do *not* require the asymmetry or transitivity of the agents' preferences. By contrast, two of the most studied random assignment rules, PS and RSD, are only defined for transitive preferences and many axiomatic results concerning these rules even require linear preferences.

A number of interesting questions arise from our study. Two of the most important ones are whether there always exists a weakly SD-envy-free popular random assignment and whether there exists a popular random assignment rule that satisfies weak SD-strategyproofness.

Acknowledgments

This material is based upon work supported by the Australian Government's Department of Broadband, Communications and the Digital Economy, the Australian Research Council, the Asian Office of Aerospace Research and Development through grant AOARD-124056, and the Deutsche Forschungsgemeinschaft under grants BR 2312/7-1 and BR 2312/7-2.

References

1. D. K. Abraham, R. W. Irving, T. Kavitha, and K. Mehlhorn. Popular matchings. *SIAM Journal on Computing*, 37(4):1030–1034, 2007.

2. H. Aziz, F. Brandt, and M. Brill. On the tradeoff between economic efficiency and strategyproofness in randomized social choice. In *Proc. of 12th AAMAS Conference*, pages 455–462. IFAAMAS, 2013.
3. P. Biró, R. W. Irving, and D. Manlove. Popular matchings in the marriage and roommates problems. In *Proceedings of the 7th Italian Conference on Algorithms and Complexity (CIAC)*, pages 97–108, 2010.
4. A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.
5. E. Budish, Y.-K. Che, F. Kojima, and P. Milgrom. Designing random allocation mechanisms: Theory and applications. *American Economic Review*, 2013. Forthcoming.
6. W. J. Cho. Probabilistic assignment: A two-fold axiomatic approach. Unpublished manuscript, 2012.
7. D. S. Felsenthal and M. Machover. After two centuries should Condorcet's voting procedure be implemented? *Behavioral Science*, 37(4):250–274, 1992.
8. P. C. Fishburn. Probabilistic social choice based on simple voting comparisons. *Review of Economic Studies*, 51(167):683–692, 1984.
9. D. C. Fisher and J. Ryan. Tournament games and positive tournaments. *Journal of Graph Theory*, 19(2):217–236, 1995.
10. P. Gärdenfors. Match making: Assignments based on bilateral preferences. *Behavioral Science*, 20(3):166–173, 1975.
11. A.-K. Katta and J. Sethuraman. A solution to the random assignment problem on the full preference domain. *Journal of Economic Theory*, 131(1):231–250, 2006.
12. T. Kavitha. Popularity vs maximum cardinality in the stable marriage setting. In *Proc. of 23rd SODA*, pages 123–134. ACM Press, 2012.
13. T. Kavitha, J. Mestre, and M. Nasre. Popular mixed matchings. *Theoretical Computer Science*, 412(24):2679–2690, 2011.
14. G. Kreweras. Aggregation of preference orderings. In S. Sternberg, V. Capecchi, T. Kloek, and C. Leenders, editors, *Mathematics and Social Sciences I: Proceedings of the seminars of Menthon-Saint-Bernard, France (1–27 July 1960) and of Gösing, Austria (3–27 July 1962)*, pages 73–79, 1965.
15. G. Laffond, J.-F. Laslier, and M. Le Breton. The bipartisan set of a tournament game. *Games and Economic Behavior*, 5:182–201, 1993.
16. M. Manea. Serial dictatorship and Pareto optimality. *Games and Economic Behavior*, 61:316–330, 2007.
17. R. M. McCutchen. The least-unpopularity-factor and least-unpopularity-margin criteria for matching problems with one-sided preferences. In *Proc. of 8th Latin American Conference on Theoretical Informatics (LATIN)*, volume 4957 of *LNCS*, pages 593–604, 2008.
18. H. Moulin. *Fair Division and Collective Welfare*. The MIT Press, 2003.
19. R. L. Rivest and E. Shen. An optimal single-winner preferential voting system based on game theory. In *Proc. of 3rd International Workshop on Computational Social Choice*, pages 399–410, 2010.
20. A. E. Roth. Repugnance as a constraint on markets. *Journal of Economic Perspectives*, 21(3):37–58, 2007.
21. H. P. Young. Dividing the indivisible. *American Behavioral Scientist*, 38:904–920, 1995.