When Hamilton cycles generate the cycle space of a random graph

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Rather new tributary to the sea of knowledge about hamiltonicity:

taking deep 'cuts' into $\mathfrak{H}$ (using a hamiltonicity-implying condition), then studying the graphs within the 'slice'. Finding proofs that those 'very Hamiltonian' graphs have extra properties.

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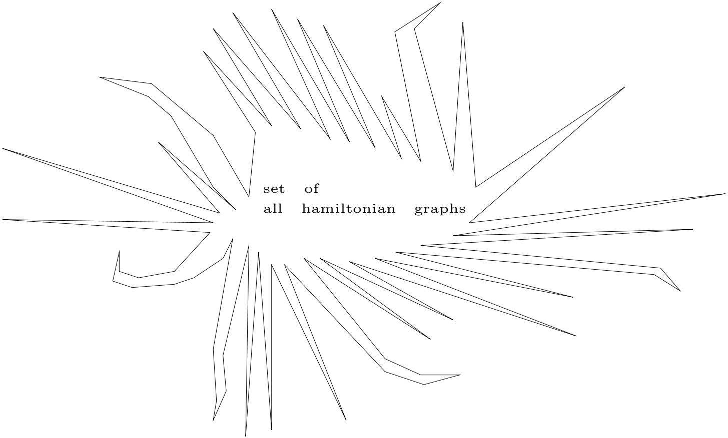

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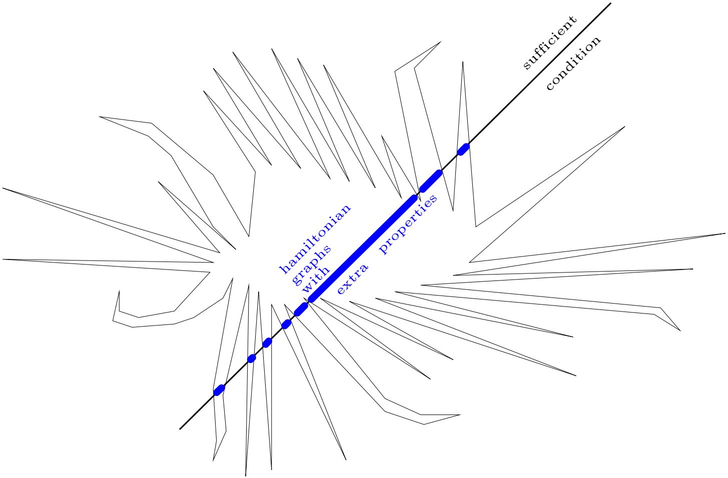

Extra property I: many Hamilton circuits

Theorem (Cuckler, Kahn 2006)

If $\delta(G) \geqslant \frac{1}{2}n$, then G contains at least $n!/(2+o(1))^n$ Hamilton circuits.

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Theorem (Glebov, Krivelevich 2012)

If $p \geqslant \frac{\log n + \log\log n + \omega(1)}{n}$, then $G_{n,p}$ a.a.s. contains at least $\sqrt{n}\left(\frac{\log n}{\mathrm{e}}(1-o(1))\right)^n$ Hamilton circuits.

random

Extra property II: many well-distributed Hamilton circuits

Theorem (D. Kühn, D. Osthus 2012)

For all $\varepsilon > 0$ and even $d \in \mathbb{N}$ there is n_0 s.t. every d-regular graph $G = (V, E)$ with $|V| \geqslant n_0$ and $d \geqslant (\frac{1}{2} + \varepsilon)|V|$ admits a decomposition of E into Hamilton circuits.

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Theorem (several papers by Frieze, Knox, Krivelevich, Kühn, Osthus, Samotij)

For any $p \in [0, 1]^{\mathbb{N}}$, a random graph $G \sim G_{n,p}$ a.a.s. contains $\lfloor \frac{1}{2}\delta(G) \rfloor$ edge-disjoint Hamilton circuits.

random

Extra property III: every cycle is symmetric difference of Hamilton circuits

Let $\mathcal{H}(G) := \{$ Hamilton circuits of $G \}$. First condition for a cycle space generated by Hamilton circuits:

Theorem (B. Alspach, S. C. Locke, D. Witte 1990)

If G is a connected Cayley graph on a finite abelian group, and if $|G|$ is odd, then $\langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} = Z_1(G; \mathbb{F}_2)$.

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First condition for arbitrary (dense) graphs:

Theorem (H. 2011)

For any $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ s.t. any graph G with odd $|G| \geqslant n_0$ and $\delta(G) \geqslant (\frac{1}{2} + \varepsilon)|G|$ has $\langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} = Z_1(G; \mathbb{F}_2)$.

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New result:

Theorem (H. 2013)

Random graphs with odd $|G|$ have $\langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} = Z_1(G; \mathbb{F}_2)$.

random

A sufficient condition for a random graph to have property $\langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} = Z_1(G; \mathbb{F}_2)$

More precisely:

Theorem (H. 2013)

If $\varepsilon > 0$ and $p \in [0,1]^{\mathbb{N}}$ with $p(n) > n^{-1/2+\varepsilon}$, then a binomial random graph $G \sim G_{n,p}$ asymptotically almost surely has the span $\langle \mathcal{H}(G) \rangle_{\mathbb{F}_2}$ as large as it can be.

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- If n is odd: every cycle is a symmetric difference of Hamilton circuits.
- If n is even:
  - every even cycle is a symmetric difference of Hamilton circuits.
  - Every cycle is a symmetric difference of Hamilton circuits and one circuit of length $n-1$.

About the proof

Essence of the proof

The set of graphs

$$\left\{ G\colon \;\; G \text{ Hamilton-connected} \quad \text{and} \quad \mathrm{Z}_1(G;\mathbb{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} \right\}$$

is a monotone graph property.

Proof of the condition $p(n) \geqslant n^{-1/2+\varepsilon}$

Theorem (D. Kühn, D. Osthus 2012)

If $p(n) \geqslant n^{-\frac{1}{2}+\varepsilon}$, then a.a.s.,
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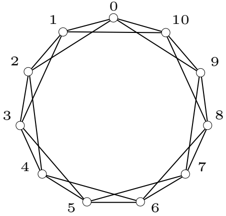

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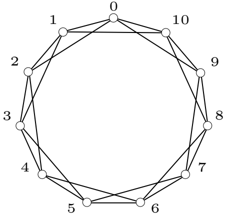

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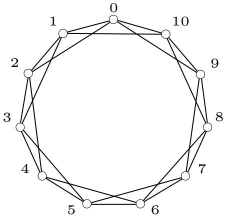
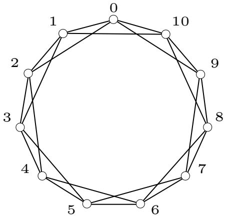


$$\in \left\{ \begin{array}{c} \text{Hamilton-connected} \\ \text{and} \quad Z_1(\cdot;\mathbb{F}_2) = \langle \mathcal{H}(\cdot) \rangle_{\mathbb{F}_2} \end{array} \right\}$$

Work in progress

Existence of C_n^2 is more than enough.
Goal: using less expensive 'rebar'.



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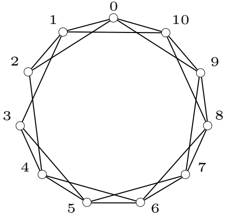

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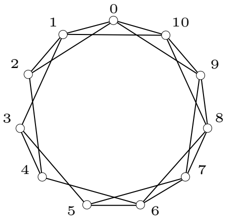

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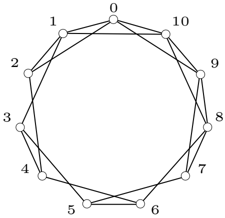

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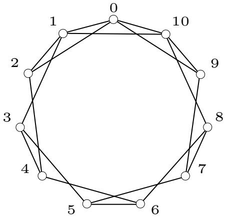

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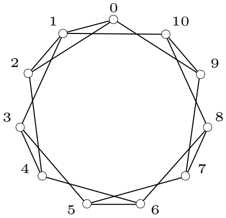

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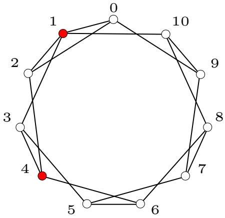

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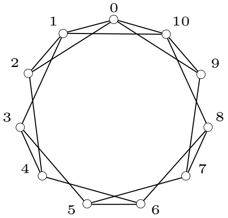
Here, we settle for:



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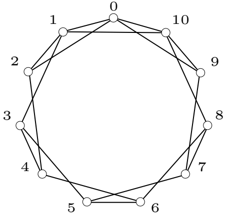
Again we need an a.a.s. embeddability-guarantee:



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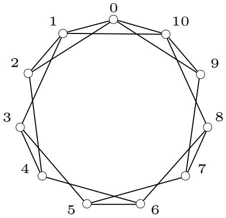

spanningly, a.a.s. → $G_{n,p}$ with $p(n) \geqslant$?

Work in progress

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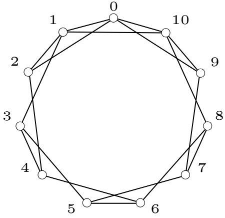
How far can the bound of Kühn and Osthus be improved for that graph?



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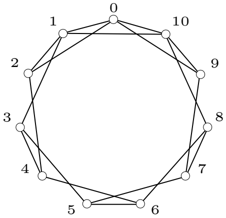
With regard to those three degree-4-vertices, you have to think big; the vast majority are degree-three vertices:



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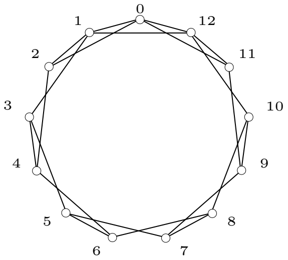

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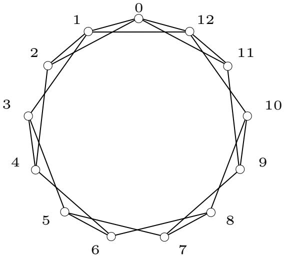
Conjecture:



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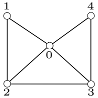
- $m \Big($ 

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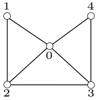
- $m\Bigg($  $\Bigg) = \color{red}{\frac{7}{5}} < \frac{3}{2}$,
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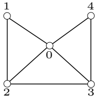
- $m\Big($ 

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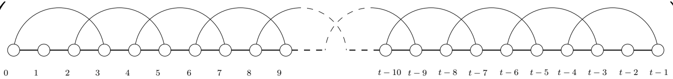
- $m^{(1)}\Big($ 

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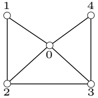
- $m\Big($ 

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Definition

Here,
(notation from the *purple book*):
$m(H) :=$
$\max\left\{\frac{\|H'\|}{|H'|} : H' \subseteq H,\ |H'| \geqslant 1\right\}$,
the parameter controlling existence of H-subgraph in $G_{n,p}$
and $m^{(1)}(H) :=$
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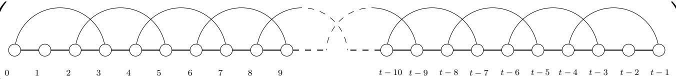
- $m^{(1)}\Big($ 

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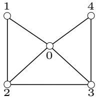
- To prove the conjecture about $p \geqslant n^{-2/3+\varepsilon}$, one seems to need one $G_{n,p}$-multi-exposure-step more than in the proof of Kühn and Osthus.
  During one exposure, the appearance of a single subgraph isomorphic to

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  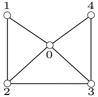

  has to be guaranteed, which is to be suitably linked to the less dense tiling pieces.

- One needs special ladder-absorbers instead of the denser $A_{j,\ell,2}$-absorbers in the proof of Kühn and Osthus.

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- However, whatever the best-possible p is, it does not coincide with the threshold for hamiltonicity:

Theorem (H. 2013)

For any $p \in [0,1]^{\mathbb{N}}$ such that a.a.s. $G_{n,p}$ is not a forest and has every cycle a symmetric difference of Hamilton circuits whenever n is odd, then we necessarily have
$p(n) > \frac{\log n + 2 \log\log n + c}{n}$
for every constant $c > 0$ and all sufficiently large odd n.

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Remark

Note we do not demand that p a.a.s. imply hamiltonian connectedness! Otherwise, the result would be obvious (if you know the threshold for minimum degree $\geqslant 3$).

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For any $p \in [0,1]^{\mathbb{N}}$ such [...] has every cycle a symmetric difference of Hamilton circuits whenever n is odd, then we necessarily have $p(n) > \frac{\log n + 2 \log\log n + c}{n}$ for every constant $c > 0$ and all sufficiently large odd n.

A challenge: does $p(n) \gg \frac{\log n + 2 \log\log n}{n}$ suffice?

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- One would have to find a 'purely probabilistic' proof, showing that $\langle \mathcal{H}(G)\rangle_{\mathbb{F}_2} = Z_1(G;\mathbb{F}_2)$ holds due to some 'global fuzzy abundance' of the Hamilton circuits, not because of the existence of one specific 'seed graph' selected in advance.

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Theorem (C. Cooper, A. M. Frieze, in Poznań 1987 conference proceedings; improvement by Cooper 1992)

If $p(n) \geqslant \frac{\log n + \log\log n + \omega(1)}{n}$, then $G_{n,p}$ a.a.s. is pancyclic.

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- Dzękuję RSA.
  Wszystkiego najlepszego z okazji trzydziesto letnich urodzin!
  Sto lat!