An asymptotic answer to a special case of an open conjecture of Bondy

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11. November 2011

A conjecture of Bondy

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$\delta(X)$:= minimum vertex degree of graph X
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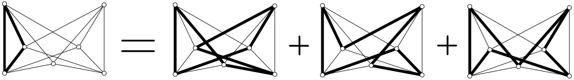

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Conjecture (A. Bondy 1979)

For every $d \in \mathbb{Z}$, in every vertex-3-connected graph X with $f_0(X) \geqslant 2d$ and $\delta(X) \geqslant d$, the set of all circuits of length at least $2d - 1$ is a $\mathbb{Z}/2$-generating system of $Z_1(X;\mathbb{Z}/2)$.

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Theorem (H. 2011)

If '$\delta(X) \geqslant d$' is replaced by '$\delta(X) \geqslant (1+\gamma)d$' for an arbitrary $\gamma > 0$, and if $f_0(X)$ is sufficiently large, then in the case of $f_0(X) = 2d$ Bondy's conjecture is true.

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Moreover, in the case of $f_0(X) = 2d$, the circuits of length $2d$ in X (Hamilton circuits) generate a codimension-1-subspace of $Z_1(X;\mathbb{Z}/2)$.

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If $f_0(X) = 2d + 1$, then of the three circuit lengths $\{f_0(X) - 2, f_0(X) - 1, f_0(X)\}$ allowed by Bondy's conjecture, $f_0(X)$ alone is enough (Hamilton circuits).

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It seems likely that the hypothesis can be weakened to a minimum degree of $150\,000\,002$, but this is not known.

The only other known sufficient condition

The first known sufficient condition for a cycle space generated by Hamilton circuits was (let $\mathcal{H}(X) := \{$ Hamilton circuits of X $\}$):

Theorem (B. Alspach, S. C. Locke, D. Witte 1990)

If X is a connected Cayley graph on a finite abelian group G, and if either $|G|$ is odd or X is bipartite, then $Z_1(X;\mathbb{Z}/2) = \langle \mathcal{H}(X) \rangle_{\mathbb{Z}/2}$.

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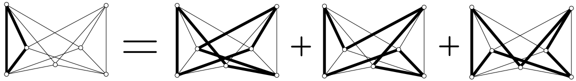
The graph underlying the following example is a 4-regular non-Cayley graph having $Z_1(X;\mathbb{Z}/2) = \langle \mathcal{H}(X) \rangle_{\mathbb{Z}/2}$:



About the proof

Essence of the proof

The set of graphs

$$\left\{ X : X \text{ Hamilton-connected} \quad \& \quad \mathrm{Z}_1(X;\mathbb{Z}/2) = \langle \mathcal{H}(X) \rangle_{\mathbb{Z}/2} \right\}$$

is a monotone graph property.

A tool in the proof

Theorem (J. Böttcher, M. Schacht, A. Taraz 2009)

$\forall\, \Delta\,:\ \forall\, \gamma > 0\,:\ \exists\, \beta > 0\,:\ \exists\, n_0\,:$
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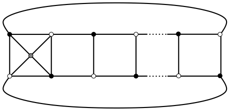

$\xhookrightarrow{\text{spanningly}}$ X with $f_0(X)$ odd
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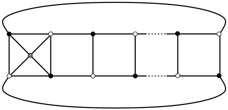

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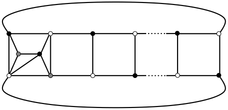

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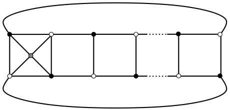

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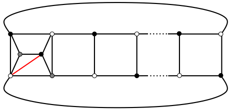

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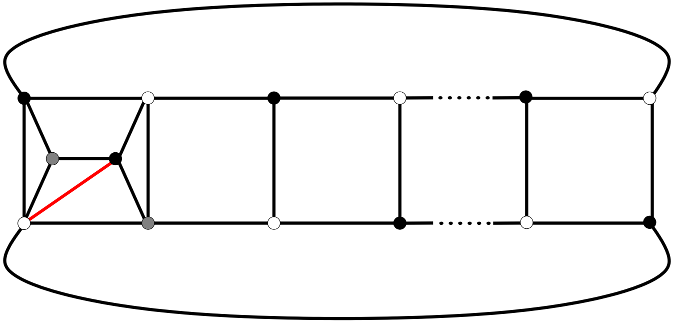

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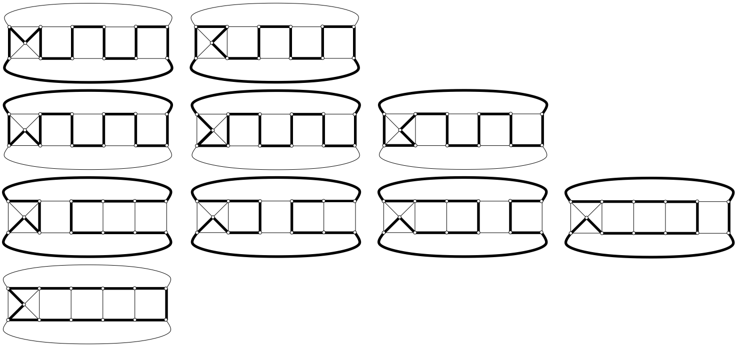

An asymptotic affirmative answer

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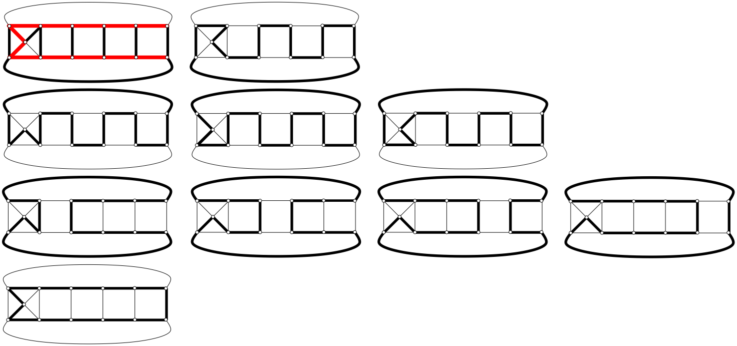

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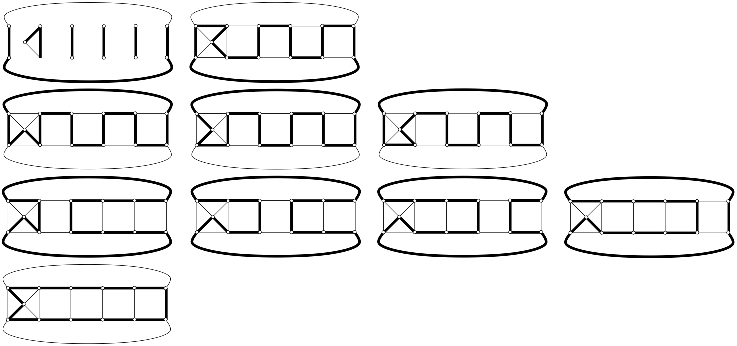

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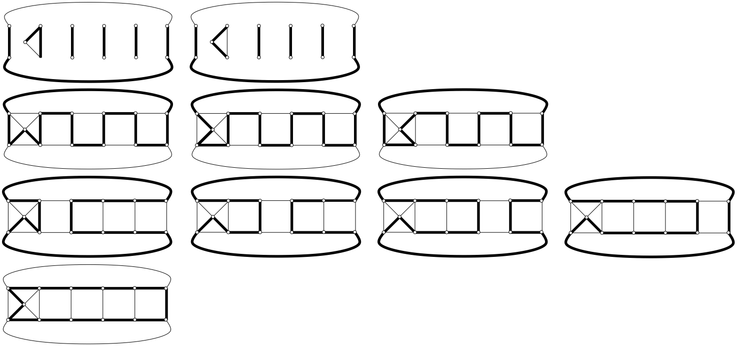

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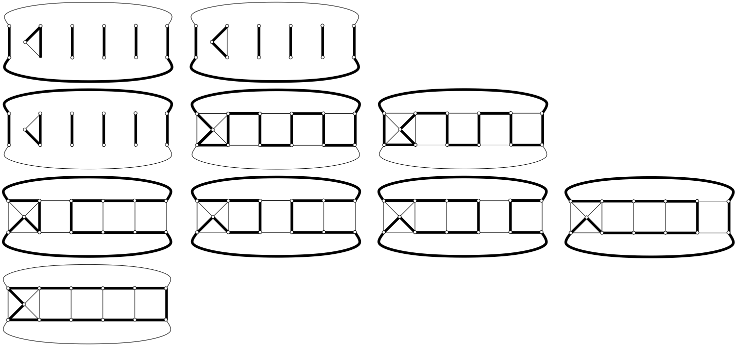

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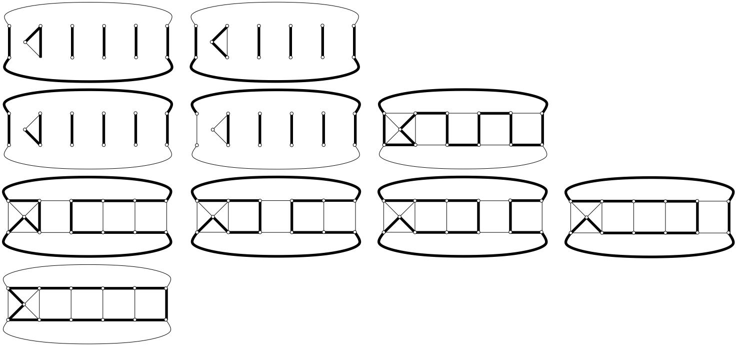

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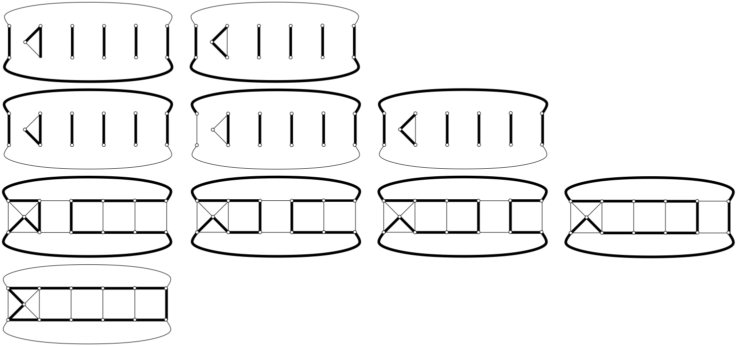

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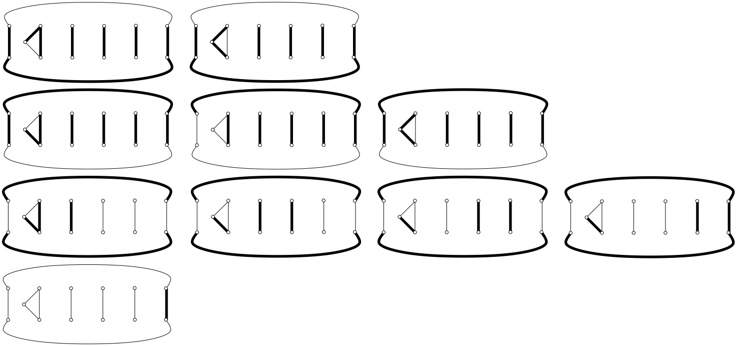

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F. Knox, D. Kühn, D. Osthus 2011:
A theorem proving a large part of the following open conjecture:

• For any p_n, an Erdős–Rényi random graph G_{n,p_n} a.a.s. realizes the obvious upper bound $\lfloor \frac{1}{2} \delta(G_{n,p_n}) \rfloor$ for the number of mutually edge-disjoint Hamilton circuits.

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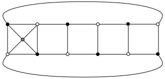
Computer experiments show that  cannot possibly serve to prove $Z_1(X; \mathbb{Z}) = \langle \mathcal{H}(X) \rangle_{\mathbb{Z}}$ in the same way as it does when proving $Z_1(X; \mathbb{Z}/2) = \langle \mathcal{H}(X) \rangle_{\mathbb{Z}/2}$.

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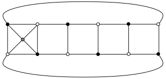
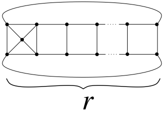
Computer experiments show that  cannot possibly serve to prove $\mathrm{Z}_1(X;\mathbb{Z}) = \langle \mathcal{H}(X) \rangle_{\mathbb{Z}}$ in the same way as it does when proving $\mathrm{Z}_1(X;\mathbb{Z}/2) = \langle \mathcal{H}(X) \rangle_{\mathbb{Z}/2}$.
One hits upon large (and odd) torsion.

If $\mathrm{Pr}_r^{\boxtimes} :=$ 

,

then $\mathrm{Z}_1(\mathrm{Pr}_r^{\boxtimes};\mathbb{Z}) / \langle \mathcal{H}(\mathrm{Pr}_r^{\boxtimes}) \rangle_{\mathbb{Z}} \cong \mathbb{Z}/(2r-1)\mathbb{Z}$.