A complete 3-dimensional Blaschke-Santaló diagram

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Abstract

Let $\mathcal{K}^n$ be the set of n-dimensional convex bodies and $K \in \mathcal{K}^n$. We study the range of values for the inradius $\mathrm{r}(K)$, circumradius $\mathrm{R}(K)$, diameter $\mathrm{D}(K)$ and minimal width $\omega(K)$ for $K \in \mathcal{K}^2$. In other words, given the function

$$f : \mathcal{K}^2 \longrightarrow [0,1]^3 \quad \text{with} \quad f(K) := \left(\frac{\mathrm{r}(K)}{\mathrm{R}(K)}, \frac{\omega(K)}{2\mathrm{R}(K)}, \frac{\mathrm{D}(K)}{2\mathrm{R}(K)}\right),$$

we compute the *Blaschke-Santaló diagram* $f(\mathcal{K}^2)$.

1. Introduction

In [1] Blaschke asked the following question:

Problem 1 (Blaschke, 1916) *Let $K \in \mathcal{K}^3$, V be the volume, S the surface area, M the integral mean curvature and*

$$h : \mathcal{K}^3 \to [0,1]^2 \quad \text{defined by} \quad h(K) := \left(\frac{4\pi \mathrm{S}(K)}{\mathrm{M}(K)^2}, \frac{48\pi^2 \mathrm{V}(K)}{\mathrm{M}(K)^3}\right).$$

Compute $h(\mathcal{K}^3)$, also known as Blaschke diagram.

It is still a famous open problem, see [4, 6].

In [7] Santaló proposed to compute diagrams for triples of r, ω, D, R, perimeter p and area A for $K \in \mathcal{K}^2$. E.g., Santaló solved the following:

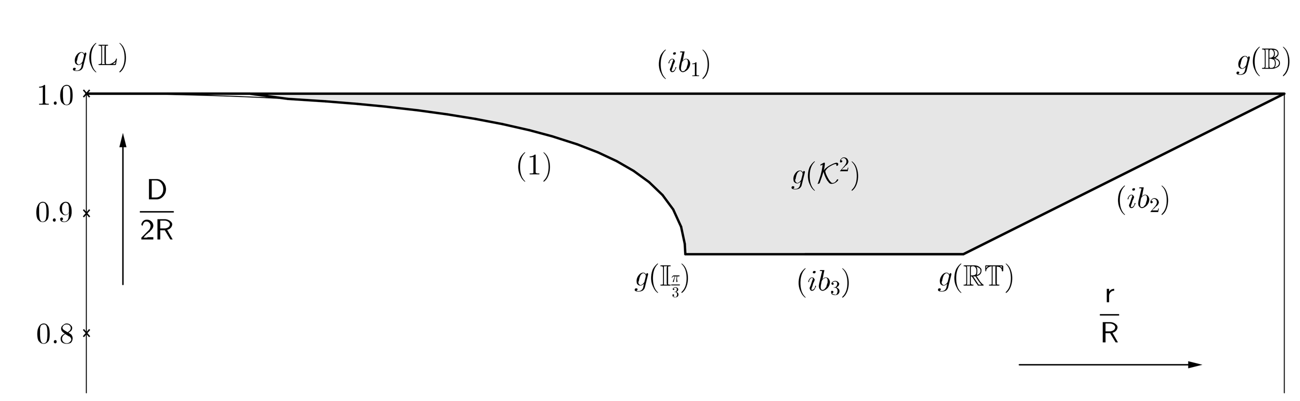
Problem 2 (Santaló, 1961) *Let $K \in \mathcal{K}^2$ and*

$$g : \mathcal{K}^2 \to [0,1]^2 \quad \text{defined by} \quad g(K) := \left(\frac{\mathrm{r}(K)}{\mathrm{R}(K)}, \frac{\mathrm{D}(K)}{2\mathrm{R}(K)}\right).$$

Compute $g(\mathcal{K}^2)$, called Blaschke-Santaló diagram.

Solving Problem 2 Santaló discovered the new inequality

$$2\mathrm{R}(K)\left(2\mathrm{R}(K) + \sqrt{4\mathrm{R}(K)^2 - \mathrm{D}(K)^2}\right)\mathrm{r}(K) \geq \mathrm{D}(K)^2\sqrt{4\mathrm{R}(K)^2 - \mathrm{D}(K)^2}. \quad (1)$$



Besides others, Hernández Cifre and Segura in [3, 5] gave full descriptions of the diagrams $\{\mathrm{r}, \omega, \mathrm{R}\}$, $\{\mathrm{r}, \omega, \mathrm{D}\}$ and $\{\omega, \mathrm{D}, \mathrm{R}\}$.

All diagrams are solved for triples of r, ω, D, R; it is a natural task to consider the Blaschke-Santaló diagram for all of them at a time.

2. Skeleton of the diagram and main results

The following Lemma is taken from [2].

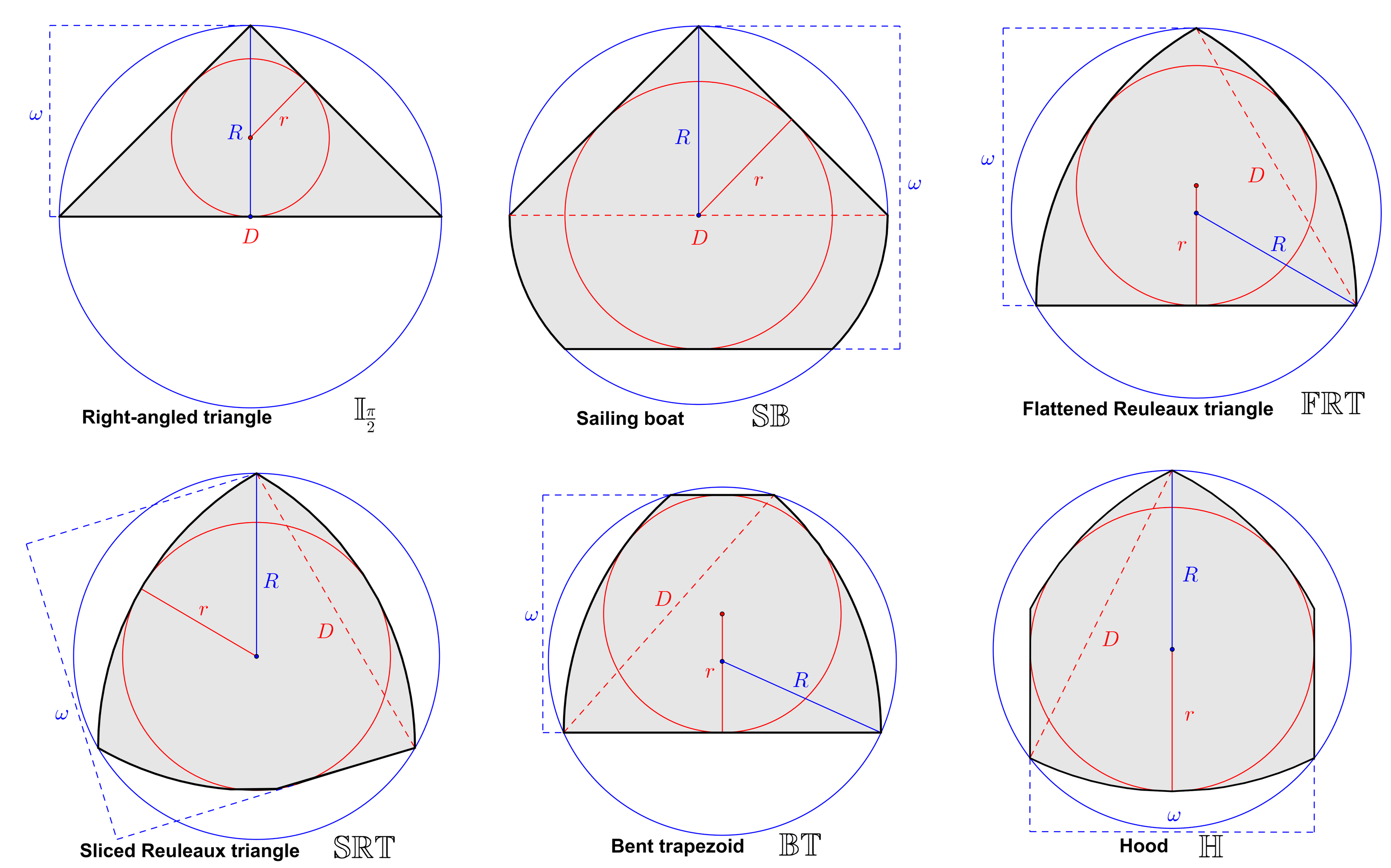
Lemma 1 $f(\mathcal{K}^2) = f(\{K \in \mathcal{K}^2 : \mathbb{B}$ *is the circumball of* $K\})$ *and* $f(\mathcal{K}^2)$ *is starshaped with respect to* $f(\mathbb{B}) = (1,1,1)$.

Thus there are *no holes* in the diagram.

The following elements form The boundary of $f(\mathcal{K}^2)$.

- *Facets*: 9 subsets of 2-dimensional differential manifolds, namely, (ub_i), (lb_i) and (ib_i) for respectively upper, lower and independent bounds of the width, $i = 1, 2, 3$.
- *Edges*: 17 subsets obtained from intersections of two facets.
- *Vertices*: 10 points obtained from the intersection of three (or more) facets.

4 "old" vertices of $g(\mathcal{K}^2)$ (line segment $\mathbb{L}$, unit ball $\mathbb{B}$, equilateral triangle $\mathbb{I}_{\frac{\pi}{3}}$ and Reuleaux triangle $\mathbb{RT}$) and 6 new:



6 of the facets are induced by already known inequalities:

Proposition 2 *Let $K \in \mathcal{K}^2$. Then*

$$2\mathrm{r}(K) \leq \omega(K) \quad (lb_1) \qquad \mathrm{D}(K) \leq 2\mathrm{R}(K) \quad (ib_1)$$

$$\omega(K) \leq \mathrm{R}(K) + \mathrm{r}(K) \quad (ub_1) \qquad \mathrm{R}(K) + \mathrm{r}(K) \leq \mathrm{D}(K) \quad (ib_2)$$

$$\sqrt{3}\mathrm{R}(K) \leq \mathrm{D}(K) \quad (ib_3) \qquad (4\mathrm{R}(K)^2 - \mathrm{D}(K)^2)\mathrm{D}(K)^4 \leq 4\omega(K)^2\mathrm{R}(K)^4 \quad (lb_2)$$

The last 3 new inequalities involve all 4 radii simultaneously:

Theorem 3 (Bent isosceles ineq.) *Let $K \in \mathcal{K}^2$. Then*

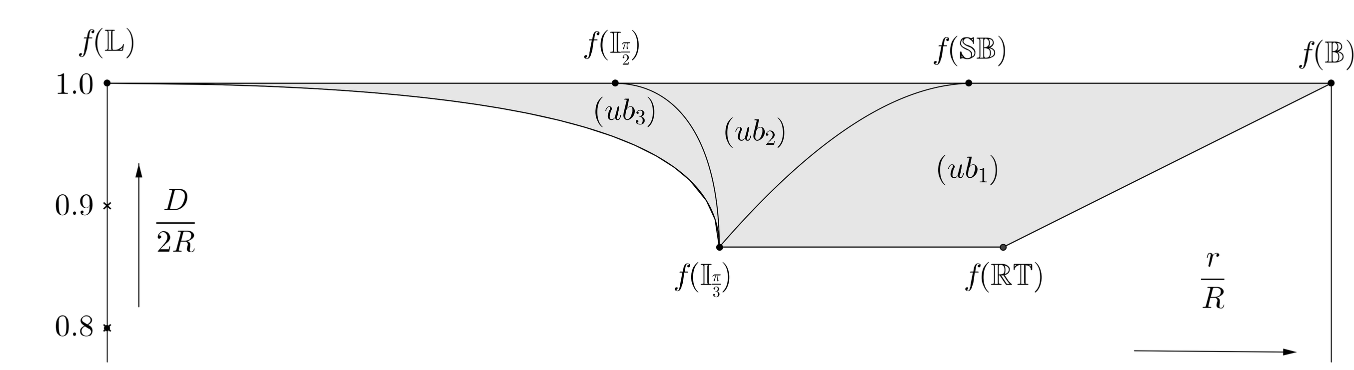
$$\omega(K) \geq 2\mathrm{D}(K)\sqrt{1 - \left(\frac{\mathrm{D}(K)}{2\mathrm{R}(K)}\right)^2} \cos\left[\arccos\left(\frac{\mathrm{D}(K)}{2(\mathrm{D}(K) - \mathrm{r}(K))}\right) + \arccos\left(\frac{\mathrm{D}(K)}{2\mathrm{R}(K)}\right) - \arcsin\left(\frac{\mathrm{r}(K)}{\mathrm{D}(K) - \mathrm{r}(K)}\right)\right] \quad (lb_3)$$

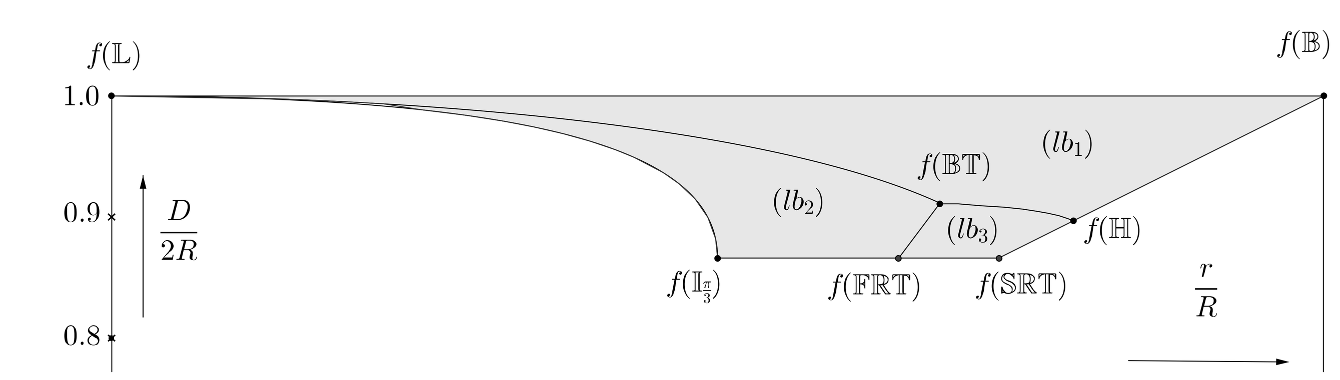
Theorem 4 (Sailing boat ineq.) *Let $K \in \mathcal{K}^2$. Then*

$$\omega(K) \leq \mathrm{r}(K)\left(1 + \frac{2\sqrt{2}\mathrm{R}(K)}{\mathrm{D}(K)}\sqrt{1 + \sqrt{1 - \left(\frac{\mathrm{D}(K)}{2\mathrm{R}(K)}\right)^2}}\right) \quad (ub_2)$$

Theorem 5 (Acute triangle ineq.) *Let $K \in \mathcal{K}^2$. Then*

$$\omega(K) \leq 2\mathrm{r}(K)\left(1 + \frac{2\mathrm{r}(K)\mathrm{R}(K)}{\mathrm{D}(K)^2}\left(1 + \sqrt{1 - \left(\frac{\mathrm{D}(K)}{2\mathrm{R}(K)}\right)^2}\right)\right) \quad (ub_3)$$





In [8] the non-complete diagram $\{\mathrm{A}, \mathrm{p}, \omega, \mathrm{D}\}$ was considered.

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