The Age of a Vertex

Tanja Gernhard

15.11.2006

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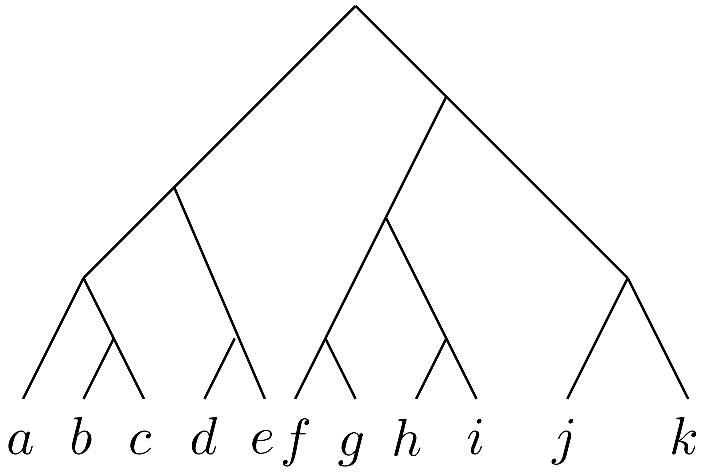

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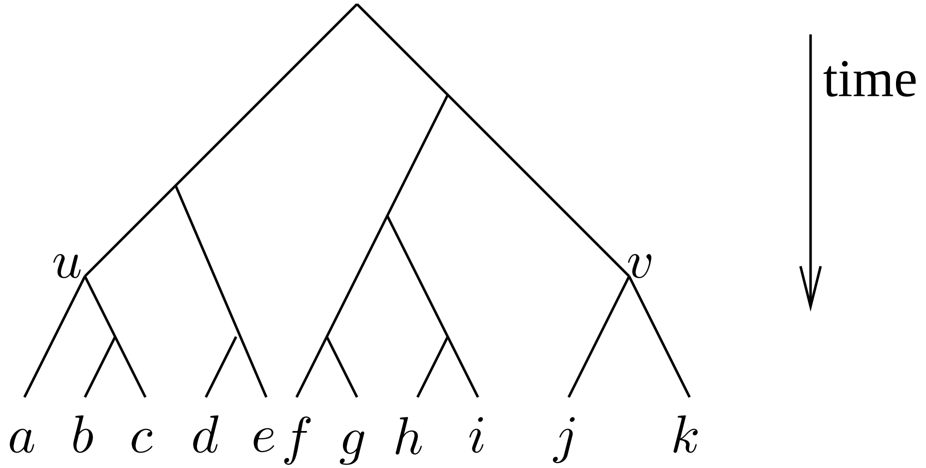

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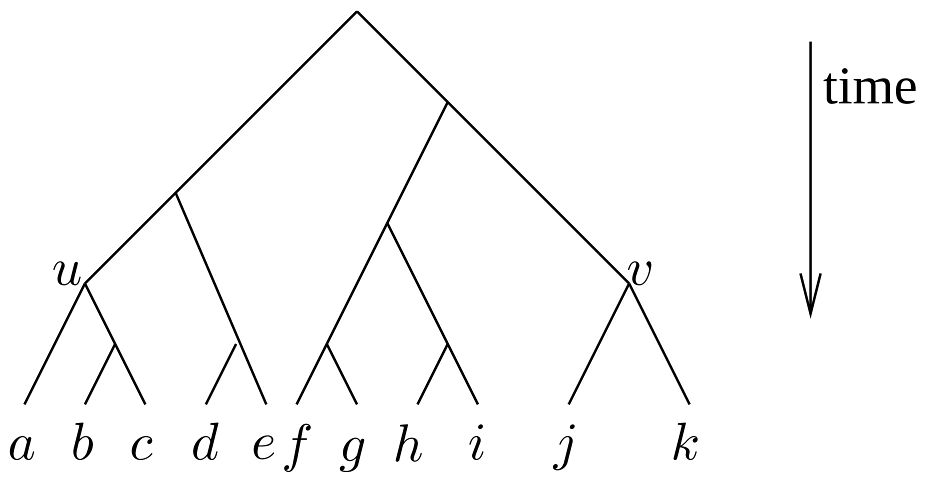
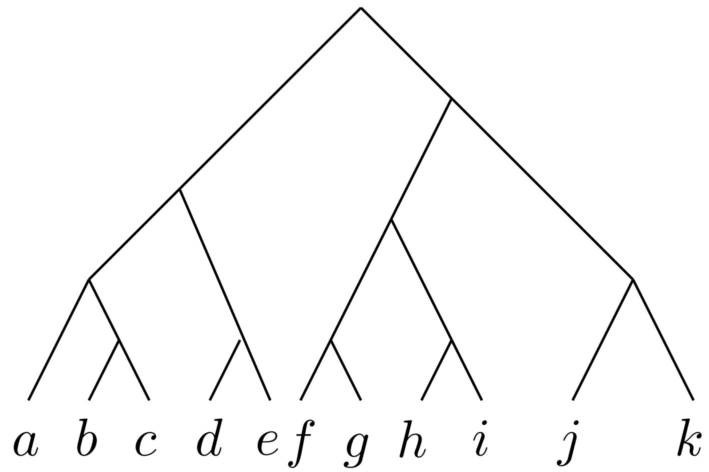
Can we say something about the relative time?

Is an inner vertex an early or late event compared to the other vertices?



The Rank Function

We want to put the inner vertices in a linear order.



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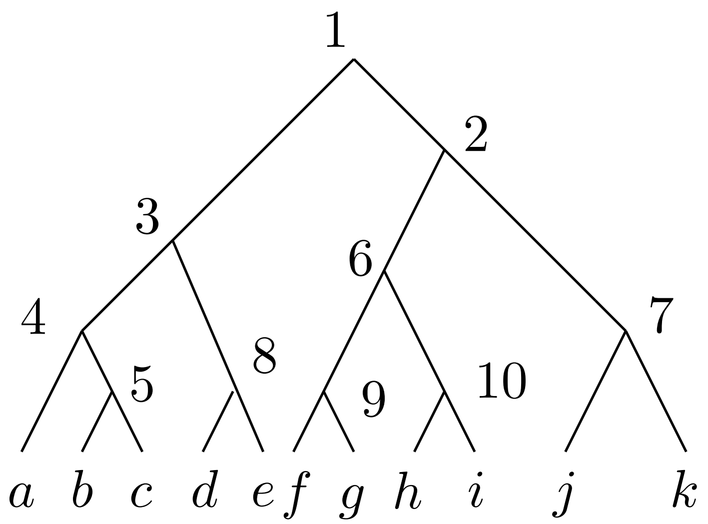

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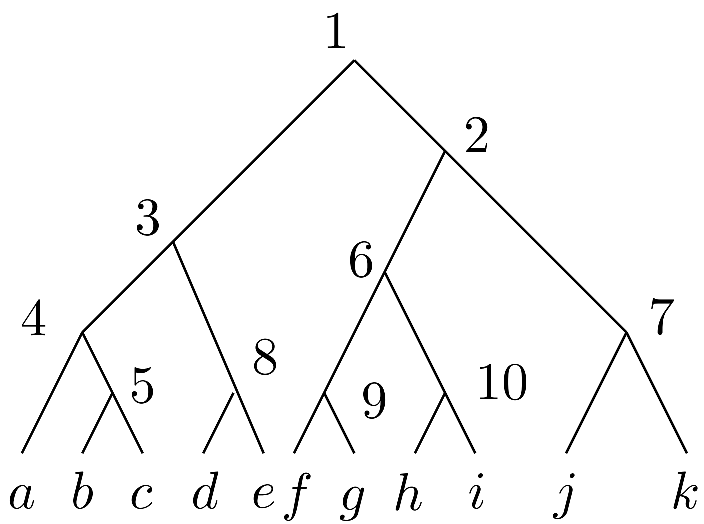
A rank function r on a tree induces a linear order on the inner vertices:

$$r : \mathring{V} \to \{1, \ldots, |\mathring{V}|\}$$

r bijection with $r(v) < r(w)$ if vertex v is an ancestor of w.



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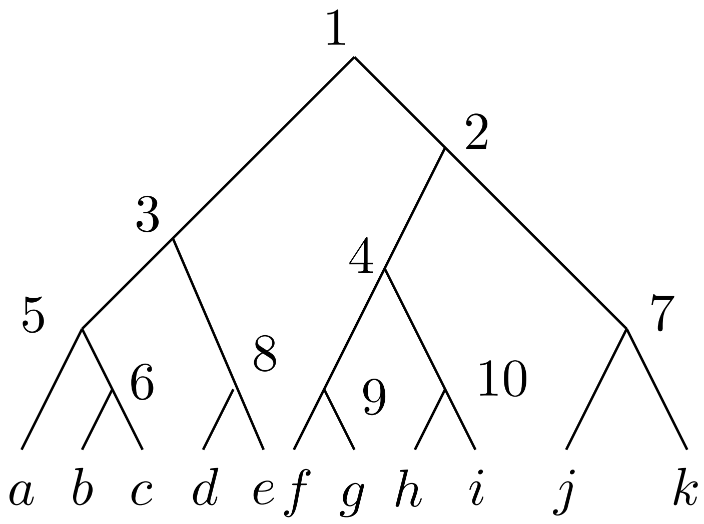

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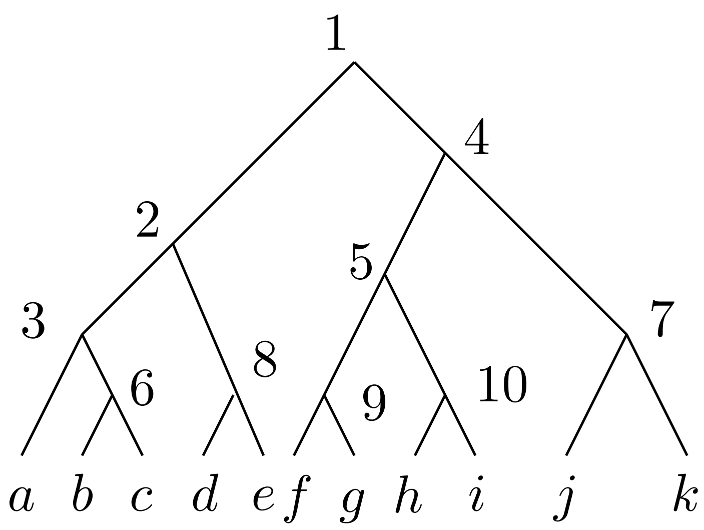

A neutral Model

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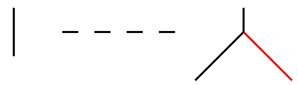

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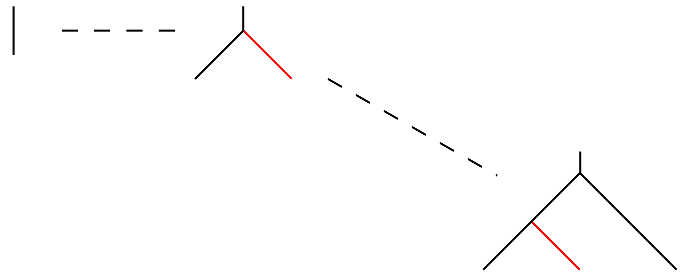

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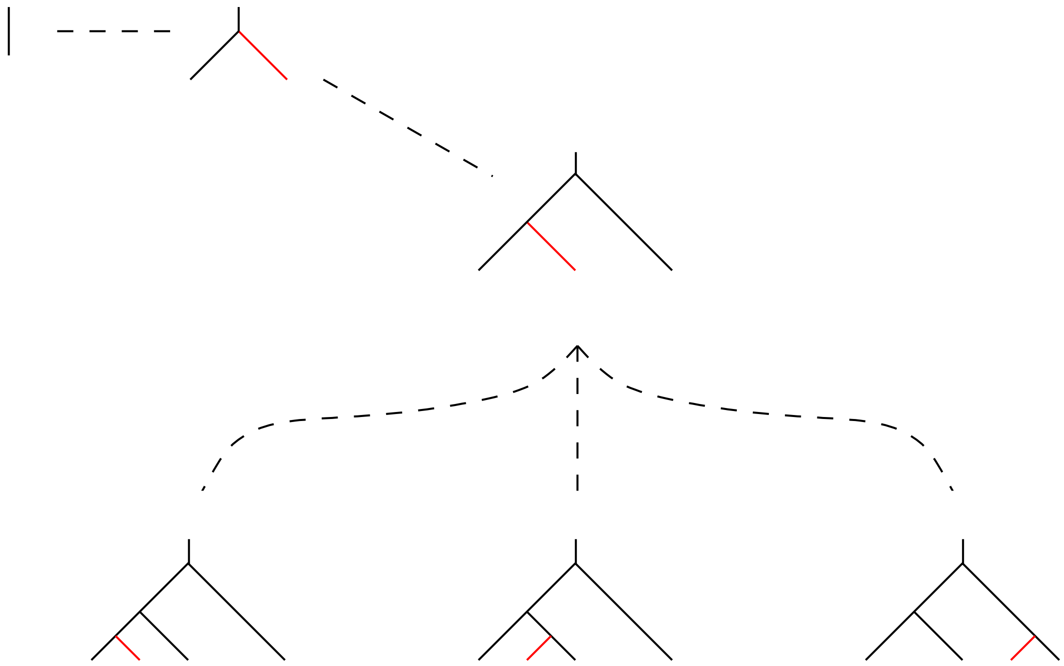
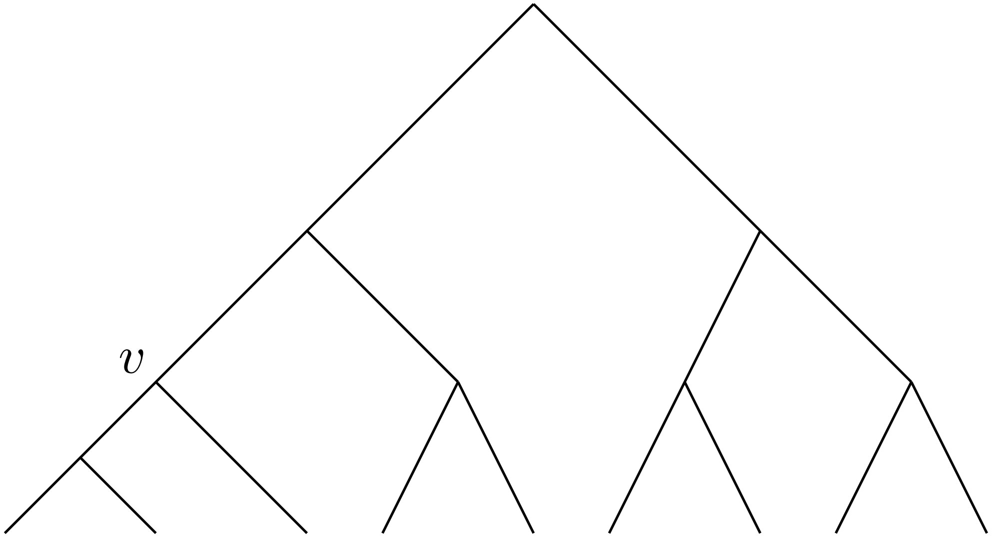


Rank of a Vertex

Under a neutral model, how likely is it that vertex v has rank 5, i.e. $r(v) = 5$?



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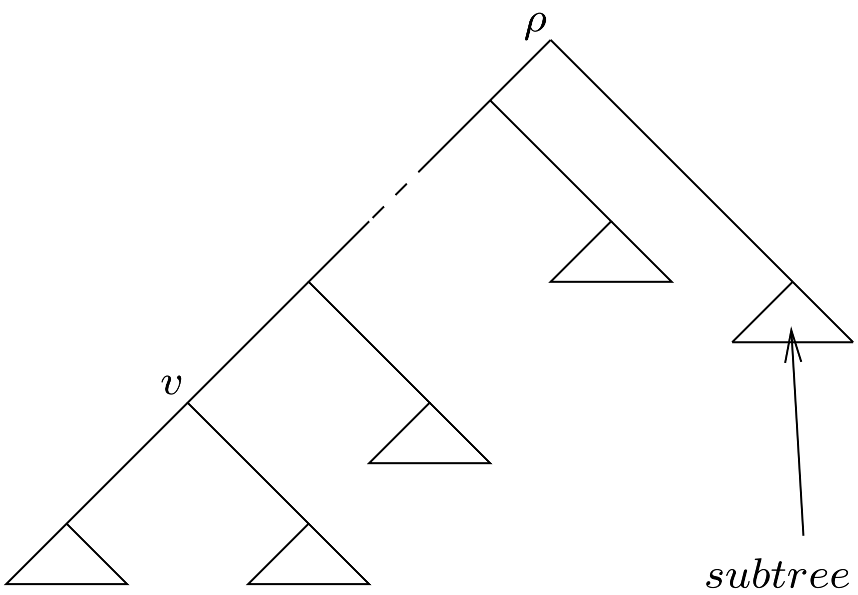

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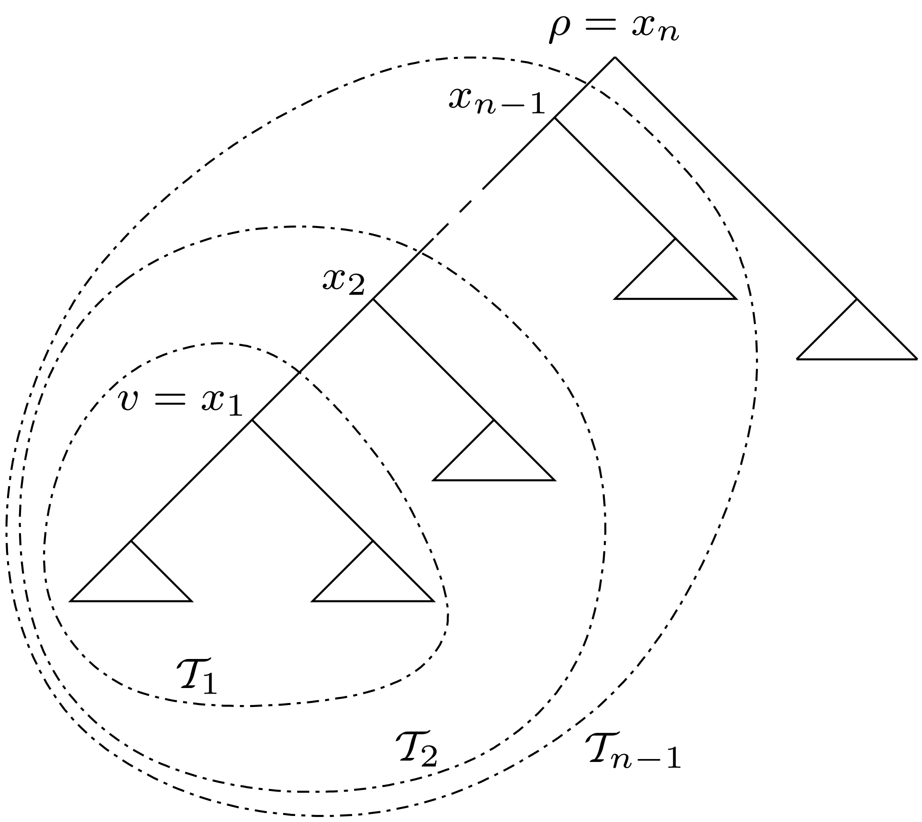
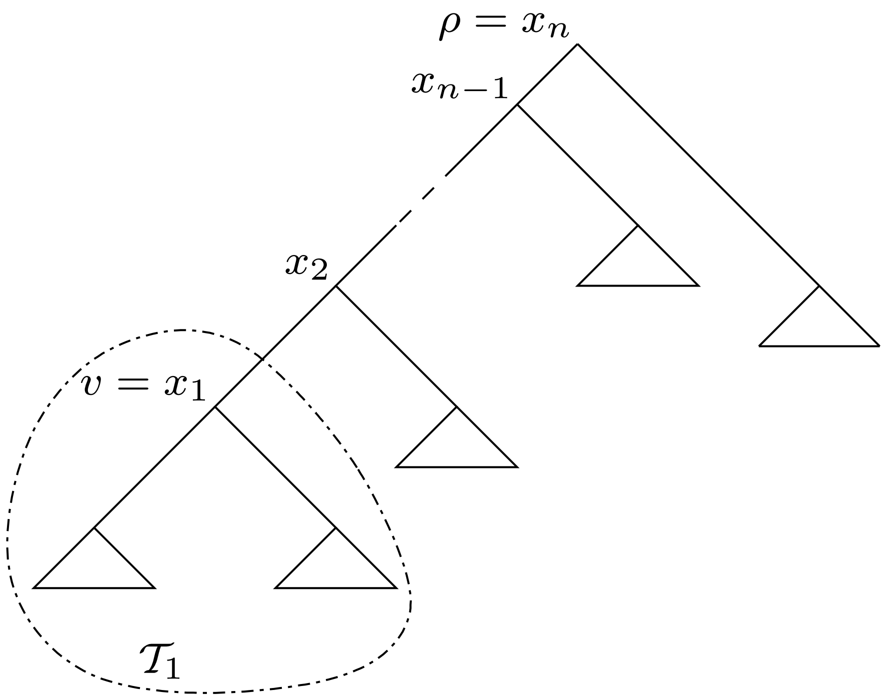
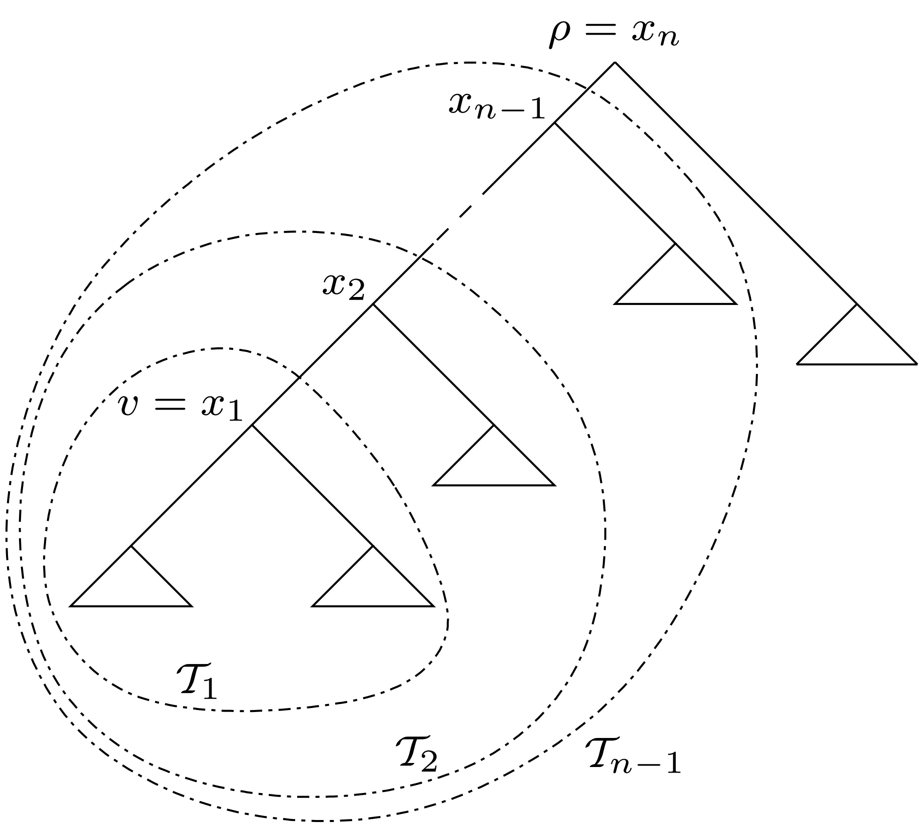


$$\alpha_{\mathcal{T}_1,v}(1) := \frac{|\mathring{V}_{\mathcal{T}_1}|!}{\prod_{v \in \mathring{V}_{\mathcal{T}_1}} \lambda_v}$$



$$\alpha_{\mathcal{T}_m,v}(i) := \sum_{j=0}^{\min\{i-2,|\mathring{V}_{\mathcal{T}''}|\}} \alpha_{\mathcal{T}',v}(i-j-1) R_{\mathcal{T}''} \binom{|\mathring{V}_{\mathcal{T}'}| + |\mathring{V}_{\mathcal{T}''}| - (i-1)}{|\mathring{V}_{\mathcal{T}''}| - j} \binom{i-2}{j}$$



Algorithm RankCountBin - *The Recursion*

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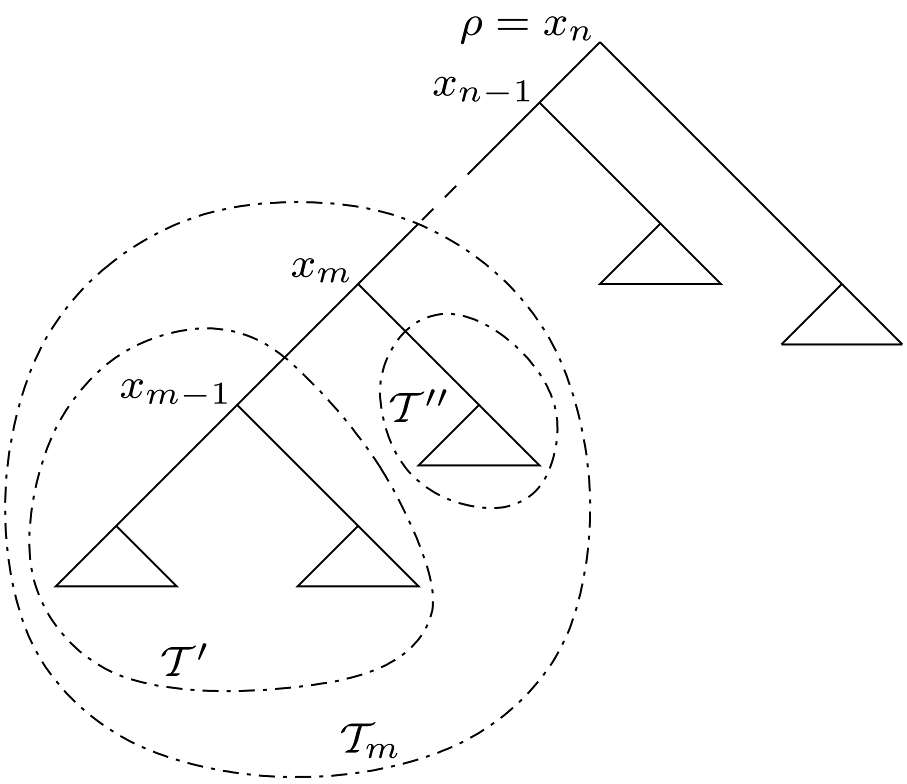

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$$\mathbb{P}[r(v) = i] = \frac{\alpha_{\mathcal{T},v}(i)}{R_{\mathcal{T}}}.$$

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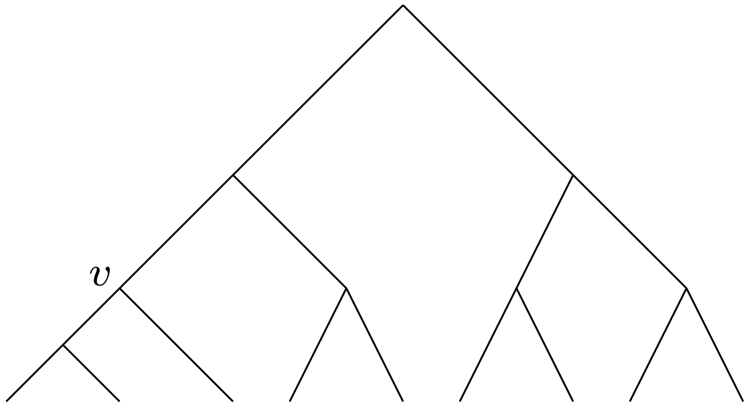
Consider the following tree:



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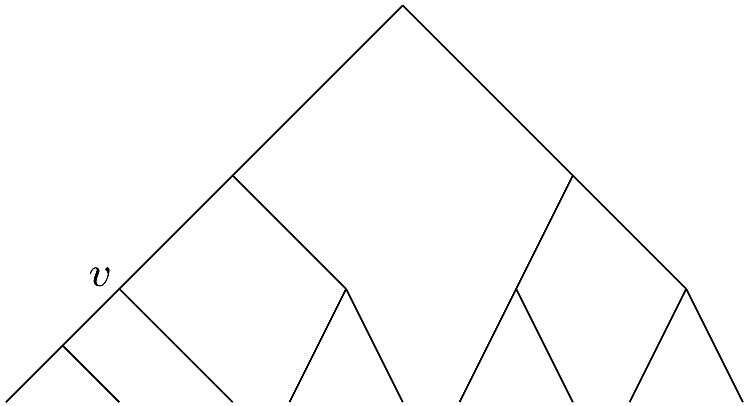

$$\begin{aligned}
\mathbb{P}[r(v)=1] &= 0\\
\mathbb{P}[r(v)=2] &= 0\\
\mathbb{P}[r(v)=3] &= \frac{20}{93}\\
\mathbb{P}[r(v)=4] &= \frac{16}{93}\\
\mathbb{P}[r(v)=5] &= \frac{27}{93}\\
\mathbb{P}[r(v)=6] &= \frac{20}{93}\\
\mathbb{P}[r(v)=7] &= \frac{10}{93}\\
\mathbb{P}[r(v)=8] &= 0
\end{aligned}$$

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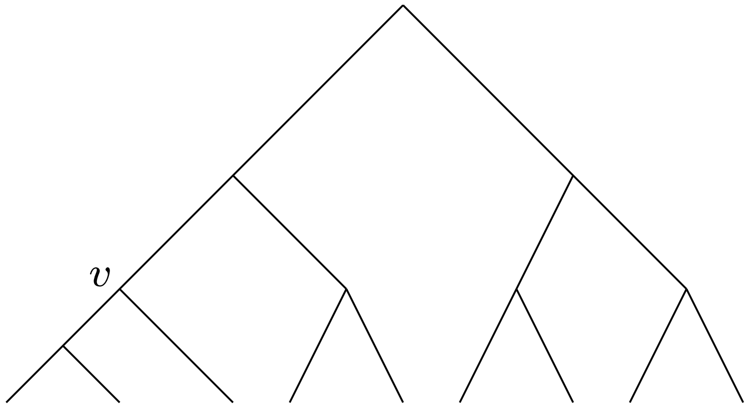

$$\mu_{r(v)} = \sum_{i=1}^{8} i\mathbb{P}[r(v) = i) \approx 4.83$$

$$\sigma_{r(v)}^2 = \sum_{i=1}^{8} i^2\mathbb{P}[r(v) = i) - \mu_{r(v)}^2 \approx 1.65$$

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  Compare runs in $O(|V|^2)$ and uses RankCountBin and RankCount.

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Let X be the random variable ‘length of edge $e = (u, v)$’ with an exp(1)-distribution (Yule model).

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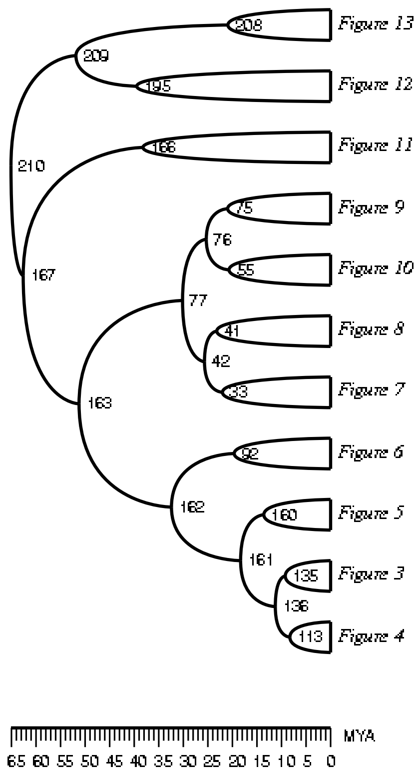
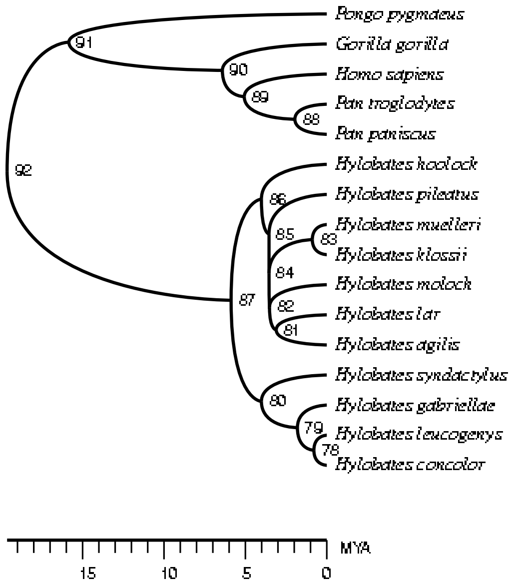
\end{aligned}$$

This expected value can be calculated with RANKCOUNT.

Edge lengths for a primate tree:





Further projects

- Include extinction into the neutral model (still uniform distribution on ranked trees for the neutral case)
- Model selection
- Parsimony network reconstruction