Technische Universität München
Zentrum Mathematik

Random Graphs (MA 5208) Summer 2012

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Sheet 1

Problem 1.1 *Properties of almost every graph.*
For two positive integers a, b we say that a graph $G = (V, E)$ is (a, b)-extendable if the following holds: for all sets $A, B \subset V$ with $|A| = a$, $|B| = b$ and $A \cap B = \emptyset$ there exists a vertex $x \in V \setminus (A \cup B)$ such that x is adjacent to all vertices in A and to none in B.

(1) Prove: for every fixed $(a, b) \in \mathbb{N}^2$, almost every graph is (a, b)-extendable.

(2) Let $k \geq 1$ be an arbitrary fixed integer. Prove or disprove:

- (i) Almost every graph is not k-connected.
- (ii) Almost every graph has minimum degree at least k.
- (iii) Almost every graph contains a separating clique.

Problem 1.2 *Large cuts in directed graphs.*
Let G denote an oriented graph (i.e. an ordinary graph together with one of two choices of an orientation for each edge). Prove that there exists a partition $A \cup B$ of $\mathrm{V}(G)$ such that the number of edges from A to B is at least $\frac{1}{4}\|G\|$.

Problem 1.3 *Square of the expectation, expectation of the square, and the second-moment method.*
Let A_n be a sequence of discrete random variables with non-negative values with respect to some probability space $\mathcal{P}$. Denote by Ex and Var the expectation and variance w.r.t. $\mathcal{P}$. Prove or disprove:
$\mathsf{Pr}[A_n \geq 1] \xrightarrow{n\to\infty} 1 \iff \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 0 \iff \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n^2]} \xrightarrow{n\to\infty} 0$

Problem 1.4 *Towards the clique number of almost every graph.*
For every $K \subset [n]$ let $\mathbf{1}_{n,K} \colon \mathcal{G}_n \to \{0, 1\}$ be defined by $\mathbf{1}_K(G) = 1 :\Leftrightarrow G[K]$ is a complete graph. Let $\mathrm{A}_{n,K} := \{\ G \in \mathcal{G}_n\colon\ G[K] \text{ is a complete graph }\}$. Let $\mathsf{Pr}_{\frac{1}{2}}$ denote the measure of $\mathrm{G}(n, \frac{1}{2})$.

(1) with $\mathrm{A}_{s,k,n} := \mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}}$, which of the three following equations is true for every $1 \leq k \leq n$:

(i) $$\sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] = \sum_{0\leq s\leq k-1} \binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s}\ \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{s,k,n}] \qquad ?$$

(ii) $$\sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] = \sum_{0\leq s\leq k-1} \binom{n}{s}\binom{\binom{n-s}{k-s}}{2}\ \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{s,k,n}] \qquad ?$$

(iii) $$\sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1}\cap \mathrm{A}_{n,C_2}] = \sum_{0\le s\le k-1} \tfrac{1}{2}\tbinom{n}{s}\tbinom{n-s}{k-s}\tbinom{n-k}{k-s}\ \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{s,k,n}] \qquad ?$$

(2) which of the three following formulas equals $\mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,[k]}\cap \mathrm{A}_{n,\{k-s+1,k-s+2,\dots,2k-s\}}]$: $(\frac{1}{2})^{(\binom{k}{2}-\binom{s}{2})}$ or $(\frac{1}{2})^{(2\binom{k}{2}-\binom{s}{2})}$ or $(\frac{1}{2})^{\binom{2k-s}{2}}$?

(3) Prove or disprove: $\binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s} = \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s}$ for every $1\le k\le n$ and every $1\le s\le k$. If you prove this, then your proof must be a 'bijective' one (not by simplifying formulas), i.e. proceed by describing two finite sets in bijective correspondence such that the left-hand side obviously counts one of them and the right-hand side the other.

(4) Let $\omega(\cdot)$ denote the clique number. Prove that $\omega(G)\ge \log|G|$ for almost every graph G.

Reminder. $[n] := \{1,\dots,n\}$, $|G|$:= number of vertices of G, $\|G\|$:= number of edges of G, $\mathcal{G}_n$:= set of all graphs on $[n]$