Sheet 1. Solutions.

Problem 1.1 *Properties of almost every graph.*
For two positive integers a, b we say that a graph $G = (V, E)$ is (a, b)-extendable if the following holds: for all sets $A, B \subset V$ with $|A| = a$, $|B| = b$ and $A \cap B = \emptyset$ there exists a vertex $x \in V \setminus (A \cup B)$ such that x is adjacent to all vertices in A and to none in B.

(1) Prove: for every fixed $(a, b) \in \mathbb{N}^2$, almost every graph is (a, b)-extendable.

(2) Let $k \geq 1$ be an arbitrary fixed integer. Prove or disprove:

  (i) Almost every graph is not k-connected.

  (ii) Almost every graph has minimum degree at least k.

  (iii) Almost every graph contains a separating clique.

A solution. As to (1), let $(a, b) \in \mathbb{N}^2$ be given. A pair $(A, B) \in \binom{[n]}{a} \times \binom{[n]}{b}$ will be called (a, b)-extendable if and only if (A, B) has the property defining '(a, b)-extendable'. Define $\psi_{a,b} \colon \mathcal{G}_n \to \mathbb{Z}_{\geq 0}$, $G \mapsto \{\ (A, B) \in \binom{[n]}{a} \times \binom{[n]}{b} \colon\ A \cap B = \emptyset \text{ and } (A, B) \text{ is not extendable in } G\ \}$. For every $(A, B) \in \binom{[n]}{a} \times \binom{[n]}{b}$ with $A \cap B = \emptyset$ define $\mathbf{1}_{A,B} \colon \mathcal{G}_n \to \{0, 1\}$ by $\mathbf{1}_{A,B}(G) = 1$ if and only if (A, B) is not extendable in G. Then $\psi_{a,b} = \sum_{(A,B) \in \binom{[n]}{a} \times \binom{[n]}{b} \colon A \cap B = \emptyset} \mathbf{1}_{A,B}$. We write $\mathsf{Pr}_{\frac{1}{2}}$ for the measure, and $\mathsf{Ex}_{\frac{1}{2}}$ for the expectation w.r.t. $\mathrm{G}(n, \frac{1}{2})$, and use the equivalent formulation '$\mathsf{Pr}_{\frac{1}{2}}[G \in \mathfrak{P}] \xrightarrow{n \to \infty} 1$' for 'almost every graph satisfies $\mathfrak{P}$'.

We have $\mathsf{Pr}_{\frac{1}{2}}[\psi_{a,b} > 0] \leq \text{(Markov)} \leq \mathsf{Ex}_{\frac{1}{2}}[\psi_{a,b}] = \binom{n}{a} \cdot \binom{n-a}{b} \cdot (1 - (\frac{1}{2})^{a+b})^{n-a-b} \xrightarrow{n \to \infty} 0$, the latter since $\binom{n}{a} \cdot \binom{n-a}{b}$ is a polynomial of fixed degree in n while $(1 - (\frac{1}{2})^{a+b})^{n-a-b} \in O(\xi^n)$ for some fixed $0 < \xi < 1$. This proves the claim.

As to (2), statement (i) is false, statement (ii) is true and statement (iii) is false.

In essence, statement (i) is false in view of (1) and because of the implication [G not k-connected] $\Longrightarrow$ [G is not $(2, k-1)$-extendable], statement (ii) is true since statement statement (i) is false, and finally statement (iii) is false because of the implication [contains a separating clique] $\Longrightarrow$ [G is not $(2, 1)$ extendable].

In detail, to prove the negation of (i), we can argue like this: assume that $G = (V, E)$ is not k-connected. By definition of k-connectedness, this means that there is a set S of $k - 1$ vertices such that $G - S$ has at least two connected components. We claim that in this case G is not $(2, k-1)$-extendable. Choose vertices a_1, a_2 in two different components of $G - S$ and set $A := \{a_1, a_2\}$ and $B := S$. Then any vertex in V which is adjacent to both vertices in A would have to lie in B and therefore we cannot find a vertex $x \in V \setminus (A \cup B)$ that satisfies the requirements needed for these sets A and B.

Hence the probability of being not k-connected is bounded from above by the probability of being not $(2, k-1)$-extendable, which tends to 0 by part (1).

To prove (ii), it suffices to note that this is implied by the negation of (i), which we have just proved.

As to (iii), let us first note that the statements (i) and (iii) do not (at any rate, not in the immediate way of one of the sets in question being a subset of the other) imply each other: if there is a separating k-set of vertices, these vertices may not be mutually adjacent; conversely, if there exists separating clique, then since nothing has been said about its size, it may consist of more than k vertices, hence its existence may not imply being not k-connected.

To use statement (1) with $a = 2$ and $b = 1$, note that

$$\begin{aligned} &\{G \in \mathcal{G}_n \colon G \text{ contains a separating clique}\} \\ =&\{G \in \mathcal{G}_n \colon \begin{array}{l}\text{there exists } S \subset [n] \text{ such that } G[S] \text{ is a complete graph} \\ \text{and } G - S \text{ has at least two connected components}\end{array} \} \\ \subset&\{G \in \mathcal{G}_n \colon G \text{ is not } (2,1)\text{-extendable}\} \quad . \end{aligned}$$

The inclusion holds since we can exhibit an offending pair (A, B) by choosing $A := \{i_1, i_2\} \in \binom{[n]\setminus S}{2}$ with i_1 and i_2 arbitrary vertices from two arbitrary connected components of $G-S$ and $B := \{b\} \in \binom{S}{1}$ with $b \in S$ arbitrary if $S \neq \emptyset$, and $b \in [n] \setminus A$ arbitary if $S = \emptyset$. No matter which $v \in [n] \setminus (A \dot\cup B)$ we now test, it cannot have the required the property: if v is in S, then it is adjacent to b, whereas if $v \in [n] \setminus S$, then if $v \in [n] \setminus S$ is in the connected components of i_1 (resp. i_2), it cannot be adjacent to i_2 (resp. i_1) by definition of 'connected component', and if $v \in [n] \setminus S$ is neither in the connected component of i_1 nor in that of i_2, it at least cannot be adjacent to both of them, again by definition of 'connected component'. Since we know by (1) that the $\mathsf{Pr}_{\frac{1}{2}}$-measure of the enclosing set converges to 0 as $n \to \infty$, this completes the proof of the negation of (iii).

Problem 1.2 *Large cuts in directed graphs.*
Let G denote an oriented graph (i.e. an ordinary graph together with one of two choices of an orientation for each edge). Prove that there exists a partition $A \dot\cup B$ of $\mathrm{V}(G)$ such that the number of edges from A to B is at least $\frac{1}{4}\|G\|$.

A solution. Given such a graph G, define the finite probability space $\mathbb{P}_G$ having sample space $W_G := \{a, b\}^{\mathrm{V}(G)}$ and each such a, b-word equally likely. For any $w \in W$ we interpret $\{i \in \mathrm{V}(G) \colon w(i) = a\}$ as A and $\mathrm{V}(G) \setminus A$ as B. For every $\{i_1, i_2\} \in \mathrm{E}(G)$ define $\mathbf{1}_{\{i_1,i_2\}} \colon W \to \{0, 1\}$ by $\mathbf{1}_{\{i_1,i_2\}}(w) = 1$ if and only if $w(i_1) = a$ and $w(i_2) = b$. Then w.r.t. $\mathbb{P}_G$ the expectation of the total number of edges of the desired type is $\mathsf{Ex}[\sum_{\{i_1,i_2\}\in\mathrm{E}(G)} \mathbf{1}_{\{i_1,i_2\}}] = \sum_{\{i_1,i_2\}\in\mathrm{E}(G)} \mathsf{Ex}[\mathbf{1}_{\{i_1,i_2\}}] = \sum_{\{i_1,i_2\}\in\mathrm{E}(G)} \mathsf{Pr}[\{w \in W \colon w(i_1) = a,\ w(i_2) = b\}] = \sum_{\{i_1,i_2\}\in\mathrm{E}(G)} \frac{1}{4} = \frac{1}{4}\|G\|$. This implies that for at least one $w \in W_G$ the partition $A \dot\cup B$ has the desired property.

Problem 1.3 *Square of the expectation, expectation of the square, and the second-moment method.*
Let A_n be a sequence of discrete random variables with non-negative values with respect to some probability space $\mathcal{P}$. Denote by Ex and Var the expectation and variance w.r.t. $\mathcal{P}$. Prove or disprove: $\mathsf{Pr}[A_n \geq 1] \xrightarrow{n\to\infty} 1 \iff \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 0 \iff \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n^2]} \xrightarrow{n\to\infty} 0$

A solution. As to $\frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 0 \iff \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n^2]} \xrightarrow{n\to\infty} 0$, this is true: on the one hand $\frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} = \frac{\mathsf{Ex}[A_n^2]-\mathsf{Ex}[A_n]^2}{\mathsf{Ex}[A_n]^2} = \frac{\mathsf{Ex}[A_n^2]}{\mathsf{Ex}[A_n]^2} - 1$, hence $\frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 0 \Leftrightarrow \frac{\mathsf{Ex}[A_n^2]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 1$. On the other hand, $\frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n^2]} =$

$\frac{\mathsf{Ex}[A_n^2]-\mathsf{Ex}[A_n]^2}{\mathsf{Ex}[A_n^2]} = 1 - \frac{\mathsf{Ex}[A_n]^2}{\mathsf{Ex}[A_n^2]}$, hence $\frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n^2]} \xrightarrow{n\to\infty} 0 \Leftrightarrow \frac{\mathsf{Ex}[A_n]^2}{\mathsf{Ex}[A_n^2]} \xrightarrow{n\to\infty} 1 \Leftrightarrow \frac{\mathsf{Ex}[A_n^2]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 1$. Taken together this proves the claimed equivalence.

As to $\mathsf{Pr}[A_n \geq 1] \xrightarrow{n\to\infty} 1 \Longleftarrow \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 0$, this is true by Thm. 1.11 (b) in the lecture of 25 april 2012 (for discrete non-negative A_n we have $\mathsf{Pr}[A_n = 0] + \mathsf{Pr}[A_n \geq 1] = 1$).

As to $\mathsf{Pr}[A_n \geq 1] \xrightarrow{n\to\infty} 1 \Longrightarrow \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 0$, this is false. Arguably the simplest example is the following artificial one: let A_n be defined by $\mathsf{Pr}[A_n = 1] := 1 - \frac{1}{n}$ and $\mathsf{Pr}[A_n = n] := \frac{1}{n}$. Then trivially $\mathsf{Pr}[A_n \geq 1] \xrightarrow{n\to\infty} 1$ since $\mathsf{Pr}[A_n \geq 1] = 1$ for every n. Yet the criterion of the second-moment method does not apply: we have $\mathsf{Ex}[A_n] = 1\cdot(1-\frac{1}{n}) + n\cdot\frac{1}{n} = 2 - \frac{1}{n}$ and $\mathsf{Ex}[A_n^2] = 1^2\cdot(1-\frac{1}{n}) + n^2\cdot\frac{1}{n} = 1 - \frac{1}{n} + n$, hence $\frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} = \frac{(1-\frac{1}{n}+n)-(2-\frac{1}{n})^2}{(2-\frac{1}{n})^2} = \frac{n-3+3n^{-1}-n^{-2}}{4-4n^{-1}+n^{-2}} \in \Omega(n)$ does not converge to 0 as $n \to \infty$.

It is of course possible to put a little probability mass on the value 0 and still preserve the mechanism of the above counterexample, so as to make the convergence $\mathsf{Pr}[A_n \geq 1] \xrightarrow{n\to\infty} 1$ a non-constant one, but this is still quite a contrived counterexample. Every known graph-theoretical counterexample for $\mathsf{Pr}[A_n \geq 1] \xrightarrow{n\to\infty} 1 \Longrightarrow \frac{\mathsf{Var}[A_n]}{\mathsf{Ex}[A_n]^2} \xrightarrow{n\to\infty} 0$ seems to be rather complex. If you find a natural graph-theoretical counterexample which can be adequately presented in roughly a page, we would be glad to hear about it.

Problem 1.4 *Towards the clique number of almost every graph.*
For every $K \subset [n]$ let $\mathbf{1}_{n,K}\colon \mathcal{G}_n \to \{0,1\}$ be defined by $\mathbf{1}_K(G) = 1 :\Leftrightarrow G[K]$ is a complete graph. Let $\mathrm{A}_{n,K} := \{\ G \in \mathcal{G}_n\colon\ G[K]$ is a complete graph $\}$. Let $\mathsf{Pr}_{\frac{1}{2}}$ denote the measure of $\mathrm{G}(n, \frac{1}{2})$.

(1.4.1) with $\mathrm{A}_{s,k,n} := \mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}}$, which of the three following equations is true for every $1 \leq k \leq n$:

(i) $$\sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] = \sum_{0\leq s\leq k-1} \binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s}\ \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{s,k,n}] \qquad ?$$

(ii) $$\sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] = \sum_{0\leq s\leq k-1} \binom{n}{s}\binom{\binom{n-s}{k-s}}{2}\ \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{s,k,n}] \qquad ?$$

(iii) $$\sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] = \sum_{0\leq s\leq k-1} \frac{1}{2}\binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s}\ \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{s,k,n}] \qquad ?$$

(1.4.2) which of the three following formulas equals $\mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}}]$: $(\frac{1}{2})^{(\binom{k}{2}-\binom{s}{2})}$ or $(\frac{1}{2})^{(2\binom{k}{2}-\binom{s}{2})}$ or $(\frac{1}{2})^{\binom{2k-s}{2}}$?

(1.4.3) Prove or disprove: $\binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s} = \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s}$ for every $1 \leq k \leq n$ and every $1 \leq s \leq k$.
If you prove this, then your proof must be a 'bijective' one (not by simplifying formulas), i.e. proceed by describing two finite sets in bijective correspondence such that the left-hand side obviously counts one of them and the right-hand side the other.

(1.4.4) Let $\omega(\cdot)$ denote the clique number. Prove that $\omega(G) \geq \log|G|$ for almost every graph G.

A solution. As to (1.4.1), only (iii) is true. To prove (iii), we note that A_{n,C_1} and A_{n,C_2} may be non-independent if $C_1 \cap C_2 \neq \emptyset$ (actually, it follows from part (1.4.2) that A_{n,C_1} and A_{n,C_2} are actually not stochastically independent if and only if $C_1 \cap C_2 \neq \emptyset$), and this suggests partitioning the index set of the left-hand sides of (i)—(iii) according to the size of $|C_1 \cap C_2|$. The notation $\{C_1, C_2\} \in \binom{\binom{[n]}{k}}{2}$ unambiguously tells us that the intersection size k does not occur. So we have

$$\sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] = \sum_{0\le s\le k-1} \sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}\colon |C_1\cap C_2|=s} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] \quad . \tag{1}$$

For any bijection $\pi\colon [n] \to [n]$ and any graph $G \in \mathcal{G}_n$ on $[n]$ define $\pi(G)$ as the graph with $\mathrm{V}(\pi(G)) := \mathrm{V}(G)$ and $\mathrm{E}(\pi(G)) := \{\{\pi(i), \pi(j)\}\colon \{i,j\} \in \mathrm{E}(G)\}$, and for any set of graphs $\mathcal{A} \subset \mathcal{G}_n$ define $\pi(\mathcal{A}) := \{\pi(G)\colon G \in \mathcal{A}\}$. The definition of $\mathrm{G}(n, \frac{1}{2})$ implies that for every bijection $\pi\colon [n] \to [n]$ and every set of graphs $\mathcal{A} \subset \mathcal{G}_n$ we have $\mathsf{Pr}_{\frac{1}{2}}[\mathcal{A}] = \mathsf{Pr}_{\frac{1}{2}}[\pi(\mathcal{A})]$. The definition of $\mathrm{A}_{n,K}$ implies that for every $\{C_1, C_2\} \in \binom{\binom{[n]}{k}}{2}$ with $|C_1 \cap C_2| = s$ and every bijection $\pi\colon [n] \to [n]$ with $\pi(C_1) = [k]$ and $\pi(C_2) = \{k-s+1, k-s+2, \ldots, 2k-s\}$ we have $\mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}} = \pi(\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2})$. The two latter statements combined imply $\mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] = \mathsf{Pr}_{\frac{1}{2}}[\pi(\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2})] = \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}}]$, which enables us to rid the inner sum in (1) of the dependence on the sets C_1 and C_2:

$$\begin{aligned} &\sum_{0\le s\le k-1} \sum_{\{C_1,C_2\}\in\binom{\binom{[n]}{k}}{2}\colon |C_1\cap C_2|=s} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}] \\ &= \sum_{0\le s\le k-1} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}}] \cdot \mathrm{k}_{k,n,s} \quad . \end{aligned} \tag{2}$$

with $\mathrm{k}_{k,n,s} := |\ \{\ \{C_1, C_2\} \in \binom{\binom{[n]}{k}}{2}\colon\ |C_1 \cap C_2| = s\ \}\ |$.

For every $0 \le s \le k-1$ we have $\binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s} = |\ \{(S, C_1'', C_2'') \in \binom{[n]}{s} \times \binom{[n]\setminus S}{k-s} \times \binom{[n]\setminus(C_1''\cup S)}{k-s}\}\ | = |\ \{(C_1', C_2') \in \binom{[n]}{k} \times \binom{[n]}{k}\colon |C_1 \cap C_2| = s\}\ |$, the latter equality due to the bijection $\{(S, C_1'', C_2'') \in \binom{[n]}{s} \times \binom{[n]\setminus S}{k-s} \times \binom{[n]\setminus(C_1''\cup S)}{k-s}\} \ni (S, C_1'', C_2'') \longmapsto (C_1''\cup S, C_2''\cup S) \in \{(C_1', C_2') \in \binom{[n]}{k} \times \binom{[n]}{k}\colon |C_1\cap C_2| = s\}$. Obviously, for every $0 \le s \le k-1$ we have $2 \cdot \mathrm{k}_{k,n,s} = |\{(C_1', C_2') \in \binom{[n]}{k} \times \binom{[n]}{k}\colon |C_1 \cap C_2| = s\}|$. This together with (2) proves (iii).

As to (1.4.2), only $(\frac{1}{2})^{2\binom{k}{2}-\binom{s}{2}}$ is the true value of $\mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}}]$, for with $E := \binom{[k]}{2} \cup \binom{k-s+1,k-s+2,\ldots,2k-s}{2}$ we have $\mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,[k]} \cap \mathrm{A}_{n,\{k-s+1,k-s+2,\ldots,2k-s\}}] = \mathsf{Pr}_{\frac{1}{2}}\ [\ \bigcap_{\{i,j\}\in E} \{G \subset \binom{[n]}{2}\colon \{i,j\} \in G\}\ ] =$ [definition of $\mathrm{G}(n, \frac{1}{2})$] $= (\frac{1}{2})^{|E|}$ and $|E| = 2\binom{k}{2} - \binom{s}{2}$. Since substituting explicit values shows that it can happen that $|\{\binom{k}{2} - \binom{s}{2}, 2\binom{k}{2} - \binom{s}{2}, \binom{2k-s}{2}\}| = 3$ and the function $x \mapsto (\frac{1}{2})^x$ is injective. Hence only $(\frac{1}{2})^{2\binom{k}{2}-\binom{s}{2}}$ is true. This completes part (1.4.2).

As to (1.4.3), this is true. Sets of the required kind are $M_{\mathrm{l}} := \{(A, B, C)\colon\ A \in \binom{[n]}{s},\ B \in \binom{[n]\setminus A}{k-s},\ C \in \binom{[n]\setminus(A\dot\cup B)}{k-s}\ \}$ and $M_{\mathrm{r}} := \{(A, B, C)\colon\ A \in \binom{[n]}{k},\ B \in \binom{A}{s},\ C \in \binom{[n]\setminus A}{k-s}\ \}$. The map $\Phi\colon M_{\mathrm{l}} \longrightarrow M_{\mathrm{r}}$, $(A, B, C) \longmapsto (A\dot\cup B, A, C)$ is evidently injective and surjective because of $\Phi(B, A\setminus B, C) = (A, B, C)$ for every given $(A, B, C) \in M_{\mathrm{r}}$. This proves (1.4.3).

As to (1.4.4), the preceding parts of the exercise were meant to suggest to you the basic setup for proving this: for every $1 \le k \le n$ let $\psi_{n,k} := \sum_{K\in\binom{[n]}{k}} \mathbf{1}_{n,K}$ (which computes the total number of k-cliques of a graph $G \in \mathcal{G}_n$). Then $\mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}] = \binom{n}{k}(\frac{1}{2})^{\binom{k}{2}}$. Let

$$\Delta := \sum_{(C_1,C_2)\in\binom{[n]}{k}\times\binom{[n]}{k}\colon C_1\ne C_2} \mathsf{Pr}_{\frac{1}{2}}[(\mathbf{1}_{n,C_1} = 1) \text{ and } (\mathbf{1}_{n,C_2} = 1)] \quad . \tag{3}$$

Then $\Delta = 2 \cdot \sum \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,C_1} \cap \mathrm{A}_{n,C_2}]$ (the sum over all $\{C_1, C_2\} \in \binom{\binom{[n]}{k}}{2}$) = (by (iii) and (1.4.2)) $= \sum_{0\leq s\leq k-1} \binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s} \left(\frac{1}{2}\right)^{2\binom{k}{2}-\binom{s}{2}}$, so $\mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}^2]$ = [Lemma 1.12 (a) in lecture of 25 april 2012] $= \mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}] + \Delta = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} + \sum_{0\leq s\leq k-1} \binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s} \left(\frac{1}{2}\right)^{2\binom{k}{2}-\binom{s}{2}}$ = [we transform according to (1.4.3), with a view towards cancellation when dividing by $\mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}]^2$] $= \sum_{0\leq s\leq k} \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s} \cdot \left(\frac{1}{2}\right)^{k(k-1)-\binom{s}{2}}$. Moreover $\mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}]^2 = \sum_{0\leq s\leq k} \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s}\left(\frac{1}{2}\right)^{k(k-1)}$, where we used the identity $\binom{n}{k} = \sum_{0\leq s\leq k} \binom{k}{s}\binom{n-k}{k-s}$ so as to make the expression for $\mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}]^2$ harmonize better with the one for $\mathsf{Ex}[\psi_{n,k}^2]$ we just found.

We now get $\mathsf{Var}_{\frac{1}{2}}[\psi_{n,k}] = \mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}^2] - \mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}]^2$ = [the summands for $s = 0$ and $s = 1$ cancel] $= \sum_{2\leq s\leq k} \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s}\left(\left(\frac{1}{2}\right)^{k(k-1)}(2^{\binom{s}{2}} - 1)\right)$ and therefore $\frac{\mathsf{Var}_{\frac{1}{2}}\,\psi_{n,k}}{\mathsf{Ex}_{\frac{1}{2}}[\psi_{n,k}]^2} = \sum_{2\leq s\leq k} \frac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} (2^{\binom{s}{2}} - 1)$ $< \sum_{2\leq s\leq k} \frac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} \cdot 2^{\binom{s}{2}} =: \beta(k,n)$. Should we be able to prove that $k_n := \lceil \log n \rceil$ implies that $\beta(n) := \beta(k_n, n) \xrightarrow{n\to\infty} 0$, then by (b) in the proof of Thm. 1.11 in the lecture of 25 april 2012 it follows that $\mathsf{Pr}_{\frac{1}{2}}[\psi_{n,k_n} \geq 1] \xrightarrow{n\to\infty} 1$, and the proof of (1.4.4) would be complete.

For every $2 \leq s \leq k$ let $\xi_{k,n,s} := \frac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} 2^{\binom{s}{2}}$, hence $\beta(k,n) = \sum_{2\leq s\leq k} \xi_{k,n,s}$.

When faced with a complicated sum involving binomial coefficients, it is sometimes[1] better to first spend some time investigating the *ratios of consecutive summands* of consecutive summand, and thereby use the good cancellation-properties of the factorial function, than to recklessly reach for ready-made estimates for binomial coefficients. Here, we will take a look at the

$$\rho_{k,n,s} := \frac{\xi_{k,n,s+1}}{\xi_{k,n,s}} = \frac{(k-s)^2}{(s+1)(n-2k+s+1)} 2^s \quad . \tag{4}$$

Lemma 1 *There exists $n_0 \in \mathbb{N}$ such that with $k_n := \lceil \log n \rceil$ we have $\rho_{k_n,n,s} < 1$ for every $n \geq n_0$ and every $2 \leq s \leq k_n$.*

Proof of Lemma 1. We prove this by making use of the $<$-preserving isomorphism of abelian groups $(\mathbb{R}_{>0}, \cdot) \xrightarrow{\log} (\mathbb{R}, +)$: the claim is equivalent to '$2\log(k_n - s) - \log(s+1) - \log(n-2k_n+s+1) + s\log(2) < 0$ for every $n \geq n_0$ and every $2 \leq s \leq k_n$', i.e. to '$h^{(1)}_{k_n,n,s} := \log(n - 2k_n + s + 1) + \log(s + 1) - (2\log(k_n - s) + s\log(2)) > 0$ for every $n \geq n_0$ and every $2 \leq s \leq k_n$'. For this it is sufficient that (statement obtained by decreasing the minuend by replacing $\log(n - 2k_n + s + 1)$ with the smaller $\log(n - 2k_n)$ and leaving out $\log(s+1)$ altogether) '$h^{(2)}_{k_n,n,s} := \log(n - 2k_n) - 2\log(k_n - s) - \log(2) \cdot s > 0$ for every $n \geq n_0$ and every $2 \leq s \leq k_n$'. Then $\frac{\mathrm{d}}{\mathrm{d}s}\, h^{(2)}_{k_n,n,s} = \frac{2}{k_n-s} - \log(2)$, hence '$\frac{\mathrm{d}}{\mathrm{d}s}\, h^{(2)}_{k_n,n,s} \substack{<\\=\\>} 0$ and $2 \leq s \leq k_n$' if and only if '$s \substack{<\\=\\>} k_n - \frac{2}{\log 2}$ and $2 \leq s \leq k_n$', hence $h^{(2)}_{k_n,n,s}$ in the interval $2 \leq s \leq k_n$ has a unique minimum at $s = k_n - \frac{2}{\log 2}$. Therefore for '$h^{(2)}_{k_n,n,s} := \log(n - 2k_n) - 2\log(k_n - s) - \log(2) \cdot s > 0$ for every $n \geq n_0$ and every $2 \leq s \leq k_n$' it is sufficient that '$h^{(3)}_n := h^{(2)}_{k_n,n,k_n-\frac{2}{\log 2}} = \log(n - 2\lceil \log n \rceil) - 2\log(\frac{2}{\log 2}) - \log(2)\lceil \log n \rceil + 2 > 0$ for every $n \geq n_0$', and for

[1] Here, for example: if we would rush to bound all binomial coefficients with e.g. the well-known rough bounds $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{\mathrm{e}\cdot n}{k}\right)^k$, there would be less cancellation in the ratio of consecutive summands than in what gives us (4). Namely, we would get $\beta(n,k) \leq \tilde{\beta}(k,n) := \sum_{2\leq s\leq k} \tilde{\xi}_{k,n,s}$ with $\tilde{\xi}_{k,n,s} := \mathrm{e}^k \left(\frac{k}{s}\right)^s \left(\frac{n-k}{k-s}\right)^{k-s} \left(\frac{k}{n}\right)^k 2^{\frac{1}{2}s^2}$, and $\frac{\tilde{\xi}_{k,n,s+1}}{\tilde{\xi}_{k,n,s}} = \frac{k}{n-k} \frac{s^s}{(s+1)^{s+1}} \frac{(k-s)^{k-s}}{(k-s-1)^{k-s-1}} 2^{s+\frac{1}{2}}$, a much more complicated expression to analyse.

this it is sufficient that (obtained by replacing $\lceil \log n \rceil$ with $\log(en) = 1 + \log n \geq \lceil \log n \rceil$) '$h_n^{(3)} \geq \log(n - 2\log(en)) - 2\log(\frac{2}{\log 2}) - \log(2)\log(en) + 2$ for every $n \geq n_0$', which via the <-preserving isomorphism $(\mathbb{R}, +) \xrightarrow{\exp} (\mathbb{R}_{>0}, \cdot)$ is equivalent to '$h_n^{(4)} := (n - 2\log(en)) \cdot (\frac{2}{\log 2})^{-2} \cdot (en)^{-\log 2} \cdot \exp(2) > 1$ for every $n \geq n_0$'. Because of $h_n^{(4)} = \exp(2)(\frac{2}{\log 2})^{-2} \cdot (\frac{1}{2} n^{1-\log(2)} - \frac{1+\log(n)}{n^{\log(2)}})$ and $(\exp(2) \cdot (\frac{2}{\log(2)})^{-2}) > 0$ and $1 - \log(2) > 0$ and the fact that n^{ε} for any fixed $\varepsilon > 0$ grows faster than $\log n$, it is now obvious that $h_n^{(4)} \xrightarrow{n\to\infty} \infty$, in particular that there exists $n_0 \in \mathbb{N}$ such that $h_n^{(4)} > 1$ for every $n \geq n_0$, completing the proof of Lemma 1.

Lemma 1 teaches us that in the sum $\beta(n) = \beta(k_n, n)$, the summands are strictly decreasing with increasing $2 \leq s \leq k_n$. Once this is known about a sum we would like to estimate, the first try is course to bound the sum by the largest summand (here, this is $\xi_{k_n,n,2}$) multiplied by the number of summands. We will now see that here, such a crude estimate is sufficient for our purposes:

$$\beta(n) = \beta(k_n, n) \leq \sum_{2 \leq s \leq k_n} \xi_{k_n,n,2} = (k_n - 1) \cdot \frac{\binom{k_n}{2}\binom{n-k_n}{k_n-2}}{\binom{n}{k_n}} < (\log n)^3 \cdot \frac{\binom{n}{k_n-2}}{\binom{n}{k_n}} = \frac{k_n(k_n-1)}{(n-k_n+2)(n-k_n+1)} =: \gamma_n \quad , \tag{5}$$

where we used $\binom{k_n}{2} < \frac{1}{2}k_n^2$ and $\binom{n-k_n}{k_n-2} < \binom{n}{k_n-2}$. Since obviously $\gamma_n \xrightarrow{n\to\infty} 0$, the proof is complete.

Remarks.

1. With hindsight the transformation of $\binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s}$ into $\binom{n}{k}\binom{k}{s}\binom{n-k}{k-s}$ is of course an avoidable detour; however it seems natural that one partitions by size of intersection and for the time being ends up with $\binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{k-s}$. For the second-moment method it can be good to first transform terms so as to make variance and square of expectation harmonize with each other, and only then plunge into estimates. This is what this was meant to illustrate.
2. With a finer analysis one can show that '$\omega(G) \geq \log|G|$ for almost every graph G' can be strengthened to the statement '$\omega(G) \geq \log_2|G| - 2\log_2\log_2\frac{1}{2}|G|$ for almost every graph G', where $\log_2$ denotes the logarithm with base 2. Even finer statements than that have been proved. However, the analysis must be done in another way. For example, Lemma 1 becomes false with $k_n := 2\log_2 n - 2\log_2\log_2\frac{n}{2} > \log n$: e.g. for $n := 10^6$ and $2 \leq s := 23 \leq k_n = 2\log_2 10^6 - 2\log_2\log_2\frac{10^6}{2} = 31.37\ldots$ we have $\rho_{2\log_2 10^6 - 2\log_2\log_2\frac{10^6}{2}, 10^6, 23} = 24.53\ldots > 1$. The statement in Problem 1.4.4 was chosen so as to be memorable and not too difficult to prove.
3. Our proof of Lemma 1 illustrates a more general technique for bounding a term involving lots of multiplicative terms ($\rho_{k,n,s}$ is such a term): you would like to prove a *stronger yet simpler* estimate by first modifying the term, and to prove your strengthened estimate you have to find an extremum. In such a situation, first transporting yourself to $(\mathbb{R}, +)$ via the <-preserving isomorphism $(\mathbb{R}_{>0}, \cdot) \xrightarrow{\log} (\mathbb{R}, +)$, doing your calculus exercises there, and perhaps switching back via $(\mathbb{R}, +) \xrightarrow{\exp} (\mathbb{R}_{>0}, \cdot)$ might save you some work since differentiation might be easier (just imagine differentiating the $\rho_{k,n,s}$, or even a simpler modification of it, with respect to the variable s, and then having to find and plug in a zero of the derivative...)
4. A remark in praise of the usefulness of the rather indirect second-moment method: by one of Bonferroni's inequalities we have

$$\begin{aligned}
& \mathsf{Pr}_{\frac{1}{2}}[\{G \in \mathcal{G}_n : \omega(G) \geq k\}] = \mathsf{Pr}_{\frac{1}{2}}\Big[\bigcup_{K \in \binom{[n]}{k}} \mathrm{A}_{n,K}\Big] \\
\text{Bonferroni's inequalities.} \quad & \geq \sum_{K \in \binom{[n]}{k}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,K}] - \sum_{\{K_1,K_2\} \in \binom{\binom{[n]}{k}}{2}} \mathsf{Pr}_{\frac{1}{2}}[\mathrm{A}_{n,K_1} \cap \mathrm{A}_{n,K_2}] \\
\text{From the same reasoning as in the solution of Problem 1.4.4.} \quad & = \binom{n}{k}(\tfrac{1}{2})^{\binom{k}{2}} - \sum_{0 \leq s \leq k-1} \binom{n}{k}\binom{k}{s}\binom{n-k}{k-s}(\tfrac{1}{2})^{(2\binom{k}{2} - \binom{s}{2} + 1)} =: \mathrm{b}_{k,n} \quad ,
\end{aligned} \tag{6}$$

and we would be done with (1.4.4) if $\mathrm{b}_{\lceil \log n \rceil, n}$ would be positive and bounded away from 0 as $n \to \infty$. Not so: we have $\mathrm{b}_{\lceil \log n \rceil, n} \xrightarrow{n\to\infty} -\infty$, hence with analytic expressions of roughly the same complexity as the input, the second-moment method synthesizes these expressions into a proof of (1.4.4) whereas head-on inclusion-exclusion truncated after two summands turns them into a useless bound.

Reminder. $[n] := \{1, \ldots, n\}$, $|G| :=$ number of vertices of G, $\|G\| :=$ number of edges of G, $\mathcal{G}_n :=$ set of all graphs on $[n]$