Sheet 2

Problem 2.1 *Quickly constructing a large network meeting given specifications.*
Suppose you need to construct (e.g. as a testing ground for some experiment) a graph G satisfying the specifications

(S.1) $|G| \geq 250$,

(S.2) $\|G\| > |G|^{1.001}$,

(S.3)  is not a subgraph of G .

You are willing to accept a chance of at most $p_{\text{tol}} := 10^{-3}$ that your construction does not satisfy the requirements. At your disposal you have a trustworthy Bernoulli(p)-source of randomness, for arbitrary p. What are you going to do? Prove that what you are planning to do meets the requirements. (Try to exercise some care to make economical choices for the parameters, in particular try to make your $|G| \geq 250$ not too large. However, this exercise does not ask you to find the smallest $|G|$ which works. Do not spend much time optimizing your estimates.)

A solution. You can take a random sample w.r.t. $\mathrm{G}(n, p_n)$. When doing so, you have to calibrate the n and p_n according to the requirements (S.1)–(S.3), using what you know from the lectures on random graphs.

We will show that choosing

$$n \geq n_0 := 663\,845 \qquad \text{and} \qquad p_n := \tfrac{1}{3} \cdot n^{-0.81} \quad , \tag{1}$$

is sufficient. For example you can take $n := 663845$ and $p := 6.41 \cdot 10^{-6}$.

In view of the definition of $\mathrm{G}(n, p_n)$, requirement (S.1) does not depend on probability, therefore does not impose a restriction on p_n, and will be met deterministically by our choice of $|G|$.

We will choose $n = |G|$ large enough to make the probability that (S.2) and (S.3) are not simultaneously satisfied smaller than p_{tol}. By the union-bound $\Pr[\bigcup A_i] \leq \sum \Pr[A_i]$, for this it is sufficient that

$$p_{\neg(\text{S.2})} + p_{\neg(\text{S.3})} \leq p_{\text{tol}} = 10^{-3} \quad , \tag{2}$$

The requirement that (S.2) be not met with a probability of at most $p_{\neg(\text{S.2})}$ implicitly imposes a lower bound on the feasible p_n.

The requirement that (S.3) be not met with a probability of at most $p_{\neg(\text{S.3})}$ implicitly imposes an upper bound on the feasible p_n.

As to (S.2), you can use the fact that $\|\cdot\|$ is sharply concentrated around its expectation. While (exponentially) stronger estimates exist (which would allow us to construct a denser graph), let us make do with a Chebyshev-derived inequality (cf. the proof of Prop. 1.13 in the lecture of

25 april 2012). Writing $ij := \{i,j\}$ and defining $\mathbf{1}_{ij}\colon \mathcal{G}_n \to \{0,1\}$ by $\mathbf{1}_{ij}(G) := 1$ if and only if $ij \in \mathrm{E}(G)$ we have $\|\cdot\| = \sum_{ij\in\binom{[n]}{2}} \mathbf{1}_{ij}$ with $\{\mathbf{1}_{ij}\colon ij \in \binom{[n]}{2}\}$ independent Bernoulli(p_n)-distributed random variables, hence $\mathsf{Ex}[\|\cdot\|] = \binom{n}{2}p_n$ and $\mathsf{Var}[\|\cdot\|] = \sum_{ij} \mathsf{Var}[\text{Bernoulli}(p_n)] = \binom{n}{2} \cdot p_n(1-p_n)$.

Abbreviate $m := \|G\|$ and $n := |G|$. We have $p_{\neg(\text{S.2})} = \mathsf{Pr}[\neq (\text{S.2})] = \mathsf{Pr}[\{G \in \mathcal{G}_n\colon m \leq n^{1.001}\}] = \mathsf{Pr}[\ \{\ G \in \mathcal{G}_n\colon\ m - \mathsf{Ex}[\|\cdot\|] \leq n^{1.001} - \mathsf{Ex}[\|\cdot\|]\ \}\ ] = \mathsf{Pr}[\ \{\ G \in \mathcal{G}_n\colon\ \mathsf{Ex}[\|\cdot\|] - m \geq \frac{1}{2}n(n-1)p_n - n^{1.001}\ \}\ ] \leq \mathsf{Pr}[\{G \in \mathcal{G}_n\colon\ |m - \mathsf{Ex}[\|\cdot\|]| \geq \frac{1}{2}n(n-1)p_n - n^{1.001}\ \}\ ] \leq [$ by Chebyshev's inequality, see e.g. Thm. 1.11 in lecture of 25 april 2012, the application of which is valid under the assumption that $\frac{1}{2}n(n-1)p_n - n^{1.001} > 0$, which we will later show to hold $] \leq \frac{\mathsf{Var}[\|\cdot\|]}{(\frac{1}{2}n(n-1)p_n - n^{1.001})^2} = \frac{\frac{1}{2}n(n-1)p_n(1-p_n)}{(\frac{1}{2}n(n-1)p_n - n^{1.001})^2}$. For further reference

$$p_{\neg(\text{S.2})} \leq \frac{\frac{1}{2}n(n-1)p_n(1-p_n)}{(\frac{1}{2}n(n-1)p_n - n^{1.001})^2} \qquad \text{if } \tfrac{1}{2}n(n-1)p_n - n^{1.001} > 0 \quad . \tag{3}$$

As to (S.3), we will describe in general how to apply the first-moment method to deal with such a restriction, then specialize to the specific forbidden subgraph. So suppose for the moment that (S.3) would be replaced by 'F is not a subgraph of G' with F an arbitrary fixed unlabelled graph.

Applying the first-moment method involves counting the number of occurrences of a substructure by indicator variables, summed over all possible 'labelled instantiations' of the substructure. In the present situation, this leads us to the following definitions: for every fixed labelled graph H with $\mathrm{V}(\mathrm{H}) = \{1, 2, \ldots, |\mathrm{H}|\}$, and for every $n \in \mathbb{N}$ and every $U \in \binom{[n]}{|\mathrm{H}|}$ define ('Sco' for 'spanning copy')

$$\mathrm{Sco}_{\mathrm{H}}(U) := \{H \subset \binom{U}{2}\colon (U, H) \text{ is isomorphic as a graph to } \mathrm{H} = (\mathrm{V}(\mathrm{H}), \mathrm{E}(\mathrm{H}))\} \quad , \tag{4}$$

and for every $H \in \mathrm{Sco}_{\mathrm{H}}(U)$ define $\mathbf{1}_{U,H}\colon \mathcal{G}_n \to \{0,1\}$ by $\mathbf{1}_{U,H}(G) := 1 :\Leftrightarrow H \subset G$. Finally define $\#_{\mathrm{H}} := \sum_{U\in\binom{[n]}{|\mathrm{H}|}} \sum_{H\in\mathrm{Sco}_{\mathrm{H}}(U)} \mathbf{1}_{U,H}$, which is a map $\mathcal{G}_n \to \{0\} \cup \mathbb{N}$.

Now take H to be any labelling of F. In order to show (this is our goal) that $\#_{\mathrm{H}}$ is very likely to be zero (i.e. there does not exists a copy of F), we use linearity of expectation to calculate

$$\begin{aligned} \mathsf{Ex}[\#_{\mathrm{H}}] &= \sum_{U\in\binom{[n]}{|\mathrm{H}|}} \sum_{H\in\mathrm{Sco}_{\mathrm{H}}(U)} \mathsf{Pr}[\{G \in \mathcal{G}_n\colon H \subset G\}] = \sum_{U\in\binom{[n]}{|\mathrm{H}|}} \sum_{H\in\mathrm{Sco}_{\mathrm{H}}(U)} p_n^{\|\mathrm{H}\|} \\ &= p_n^{\|\mathrm{H}\|} \sum_{U\in\binom{[n]}{|\mathrm{H}|}} \sum_{H\in\mathrm{Sco}_{\mathrm{H}}(U)} 1 = p_n^{\|\mathrm{H}\|} \cdot \binom{n}{|\mathrm{H}|} \cdot |\mathrm{Sco}_{\mathrm{H}}(U)| =: \mathrm{b}_{\mathrm{H}}(n) \quad . \end{aligned} \tag{5}$$

Because of $\mathsf{Pr}[\#_{\mathrm{H}} > 0] \leq \mathsf{Ex}[\#_{\mathrm{H}}]$, it is sufficient to choose n and p_n in such a way that $p_{\neg(\text{S.2})} + \mathrm{b}_{\mathrm{H}}(n) < p_{\mathrm{tol}}$. We are thus led[1] to the task of determining $|\mathrm{Sco}_{\mathrm{H}}(U)|$:

Lemma 1 *Let U be a finite set, let* H *be a graph with vertex set U. Then* $|\mathrm{Sco}_{\mathrm{H}}(U)| = \frac{|U|!}{|\mathrm{Aut}(\mathrm{H})|}$.

Proof of Lemma 1. For a set U denote by $\mathfrak{S}(U)$ the set of all permutations of U. For every $\pi \in \mathfrak{S}(U)$ define the bijection $\alpha(\pi)\colon \mathrm{Sco}_{\mathrm{H}}(U) \to \mathrm{Sco}_{\mathrm{H}}(U)$ by $H \mapsto \alpha(\pi)(H) := \{\{\pi(i), \pi(j)\}\colon \{i,j\} \in H\}$. This defines a group action $\alpha\colon \mathfrak{S}(U) \to \mathfrak{S}(\mathrm{Sco}_{\mathrm{H}}(U))$. The group $\mathfrak{S}(U)$ via α acts transitively on

[1]If the only interest would be the convergence $\mathrm{b}_{\mathrm{H}}(n) \xrightarrow{n\to\infty} 0$, then one could just say that $|\mathrm{Sco}_{\mathrm{H}}(U)|$ is a constant independent of n. If completely explicit estimates are called for, one needs to know this number.

$\mathrm{Sco_H}(U)$, i.e. for any $H_{\mathrm{rep}} \in \mathrm{Sco_H}(U)$ for its orbit we have $\{\alpha(\pi)(H_{\mathrm{rep}})\colon \pi \in \mathfrak{S}(U)\} = \mathrm{Sco_H}(U)$. With the abbreviation $\mathrm{Stab}^{\alpha}_{\mathfrak{S}(U)}(H_{\mathrm{rep}}) := \{\pi \in \mathfrak{S}(U)\colon \alpha(\pi)(H_{\mathrm{rep}}) = H_{\mathrm{rep}}\}$, we have $\mathrm{Stab}^{\alpha}_{\mathfrak{S}(U)}(H_{\mathrm{rep}}) = \mathrm{Aut}(H_{\mathrm{rep}}) \cong \mathrm{Aut(H)}$, the latter isomorphism by choice of H_{rep} and definition of $\mathrm{Sco_H}(U)$. Finally, by the orbit-stabilizer-formula, $|\mathrm{Sco_H}(U)| = \frac{|\mathfrak{S}(U)|}{|\mathrm{Stab}^{\alpha}_{\mathfrak{S}(U)}(H_{\mathrm{rep}})|}$. The claim of Lemma 1 follows.

Specializing the general formalism that we have set up so far to the specific forbidden subgraph in Problem 2.1, let H denote the graph with $\mathrm{V(H)} := \{1, \dots, 4\}$ and $\mathrm{E(H)} := \{\ \{1,2\},\ \{2,3\},\ \{3,4\},\ \{1,4\},\ \{1,3\}\ \}$. This H is a labelling of ⌑ and we have $|\mathrm{H}| = 4$, $\|\mathrm{H}\| = 5$ and $|\mathrm{Aut(H)}| = 4$ (by a short exhaustive inspection). Combining Lemma 1 with (5),we find

$$p_{\neg(\mathrm{S.3})} = \mathsf{Pr}[\#_{\mathrm{H}} > 0] \le \mathsf{Ex}[\#_{\mathrm{H}}] = \binom{n}{4} \cdot \tfrac{4!}{4} \cdot p_n^{\|\mathrm{H}\|} = \tfrac{1}{4}n(n-1)(n-2)(n-3)p_n^5 < \tfrac{1}{4}n^4p_n^5 \quad . \tag{6}$$

In view of (2), (3) and (6), we now know that it is sufficient to choose n and p_n in such a way that

(C.1) $\frac{1}{2}n(n-1)p_n - n^{1.001} > 0$,

(C.2) $\frac{\frac{1}{2}n(n-1)p_n(1-p_n)}{(\frac{1}{2}n(n-1)p_n - n^{1.001})^2} + \frac{1}{4}n^4p_n^5 \le 10^{-3}$.

If (1) is true, then on the one hand, since $n \ge n_0 = 663845$ and since $\frac{1}{2}n(n-1) \ge \frac{1}{2.1}n^2$ for every $n \ge 21$, we have $\frac{1}{2}n(n-1)p_n - n^{1.001} \ge \frac{1}{2.1}n^{1.19} - n^{1.001}$. So for $n \ge n_0$ we indeed have (C.1). Moreover, the function $n \mapsto \frac{1}{2.1}n^{1.19} - n^{1.001}$ is strictly increasing. It follows that for each $n \ge n_0$ we are guaranteed by (3) that $p_{\neg(\mathrm{S.3})} \le \frac{\frac{1}{2}n(n-1)p_n(1-p_n)}{(\frac{1}{2}n(n-1)p_n - n^{1.001})^2} \le \frac{\frac{1}{6}n^{1.19}(1-\frac{1}{3}n^{-0.81})}{(\frac{1}{6.3}n^{1.19} - n^{1.001})^2} \le [$ since $n \ge n_0$ and $\frac{1}{6.3}n^{1.19} - n^{1.001} \ge \frac{1}{12.6}n^{1.19}$ for every $n \ge 12.6^{1/(0.189)} = 663844.5226\dots$ $] \le \frac{\frac{1}{6}n^{1.19}(1-\frac{1}{2}n^{-0.81})}{(\frac{1}{12.6}n^{1.19})^2} \le \frac{\frac{1}{6}n^{1.19}}{(\frac{1}{12.6}n^{1.19})^2} \le [$ since $(12.6)^2/6 \le 30$ $]$ $30\ n^{-1.19} \le [$ since $n \ge n_0$ and $30\ n_0^{-1.19} = 3.5388\dots \cdot 10^{-6}$ $] \le 4 \cdot 10^{-6}$. On the other hand, by (6), we have $p_{\neg(\mathrm{S.3})} \le \frac{1}{4}n^4p_n^5 \le \frac{1}{4} \cdot n^4 \cdot \frac{1}{3^5} \cdot n^{-4.05} = \frac{1}{4\cdot 3^5}n^{-0.05} \le [$ since $n \ge n_0$ and $\frac{1}{4\cdot 3^5}n_0^{-0.05} = 5.2490\dots \cdot 10^{-4}$ $] \le 6 \cdot 10^{-4}$. Thus, if (1) is true, then both (C.1) and (C.2) are true.

For the explicit choices $n := 663845$ and $p := 6.41 \cdot 10^{-6}$ the calculation to check (C.1) and (C.2) are $\frac{1}{2}n(n-1)p_n - n^{1.001} = 739605.61\dots > 0$, and $\frac{\frac{1}{2}n(n-1)p_n(1-p_n)}{(\frac{1}{2}n(n-1)p_n - n^{1.001})^2} + \frac{1}{4}n^4p_n^5 = 2.58\dots \cdot 10^{-6} + 5.2629\dots \cdot 10^{-4} < 10^{-3} = p_{\mathrm{tol}}$.

Therefore, with these choices, the specifications are true with at most the tolerated probability of failure.

Assuming[2] that both sampling from your source of randomness and the operation of writing to memory that a 2-set is or is not an edge, take one nanosecond per operation, you can carry out the construction in $\binom{663845}{2} \cdot 2 \cdot 10^{-9}$ seconds $= 440.6\dots$ seconds, i.e. in roughly eight minutes.

Problem 2.2 *Almost always, all maximal cliques are large.*

(1) Let $s \in \mathbb{N}$. Find an example of a connected graph G which for every $2 \le i \le s$ contains an inclusion-maximal clique on i vertices.

[2] In may 2012, the English Wikipedia writes under ‘Nanosecond’: “Times of this magnitude are commonly encountered in telecommunications, pulsed lasers and some areas of electronics.”

(2) Prove that almost every graph G does not contain any inclusion-maximal clique with less than $\frac{1}{2}\log|G|$ vertices.

A solution. As to (1), an example is obtained by starting with cliques of distinct orders $2,\dots,s$, then choosing an arbitrary ordering of the cliques, and finally ensuring connectedness by connecting the cliques with one edge per pair along the chosen ordering (any choice of the vertices incident to the $s-2$ connecting edges will do; this example has $\frac{1}{2}(s^2+s-2)$ vertices and $(s-2)+\sum_{2\le i\le s}\binom{i}{2}=\frac{1}{6}(s^3+5s-12)$ edges).

As to (2), for every $K\subset[n]$ define $\mathbf{1}_K\colon\mathcal{G}_n\to\{0,1\}$ by $G\mapsto\mathbf{1}_K(G):=1:\Leftrightarrow K$ is an inclusion-maximal clique in G. Then, with Pr the measure of $\mathrm{G}(n,\frac{1}{2})$, for every $1\le r\le n$ and every $K\subset[n]$ with $|K|=r$,

$$\mathsf{Pr}[\{G\in\mathcal{G}_n\colon\mathbf{1}_K(G)=1\}]=(\tfrac{1}{2})^{\binom{r}{2}}\cdot(1-(\tfrac{1}{2})^r)^{n-r}=\frac{(2^r-1)^{n-r}}{2^{rn-\frac{1}{2}r^2-\frac{1}{2}r}}\quad. \tag{7}$$

Let $b_n:=\lfloor\frac{1}{2}\log n\rfloor$ and let $\Psi:=\sum_{1\le r\le b_n}\sum_{K\in\binom{[n]}{r}}\mathbf{1}_K$. Then $\Psi(G)$ is the total number of inclusion-maximal cliques on at most b_n vertices in a $G\in\mathcal{G}_n$. With $\sigma_{r,n}:=\binom{n}{r}\cdot\frac{(2^r-1)^{n-r}}{2^{rn-\frac{1}{2}r^2-\frac{1}{2}r}}$ we employ Markov's inequality and (7) to find $\mathsf{Pr}[\Psi\ge1]\le\mathsf{Ex}[\Psi]=\Phi_n:=\sum_{1\le r\le b_n}\sigma_{r,n}$. Problem 2.2(2) would be finished if we could show that $\Phi_n\xrightarrow{n\to\infty}0$ (strictly speaking, we would then have proved the slightly stronger statement obtained by replacing 'less than' with 'at most'). Due to the bound $\binom{n}{r}\le(\frac{\mathrm{e}n}{r})^r$, showing $\Phi'_n:=\sum_{1\le r\le b_n}\sigma'_{r,n}\xrightarrow{n\to\infty}0$, where $\sigma'_{r,n}:=(\frac{\mathrm{e}n}{r})^r\cdot\frac{(2^r-1)^{n-r}}{2^{rn-\frac{1}{2}r^2-\frac{1}{2}r}}$, is sufficient for showing $\Phi_n\xrightarrow{n\to\infty}0$. Obviously, each of the first five summands of Φ'_n, namely $\sigma'_{1,n}:=\mathrm{e}n\frac{1}{2^{n-1}}$, $\sigma'_{2,n}:=(\frac{\mathrm{e}n}{2})^2\,\frac{3^{n-2}}{2^{2n-3}}<(\frac{\mathrm{e}n}{2})^2\,(\frac{3}{4})^{n-2}$, $\sigma'_{3,n}:=(\frac{\mathrm{e}n}{3})^3\,\frac{7^{n-3}}{2^{3n-6}}<(\frac{\mathrm{e}n}{2})^3\,(\frac{7}{8})^{n-3}$, $\sigma'_{4,n}:=(\frac{\mathrm{e}n}{4})^4\frac{15^{n-4}}{2^{4n-10}}<(\frac{\mathrm{e}n}{4})^4\,(\frac{15}{16})^{n-4}$, $\sigma'_{4,n}:=(\frac{\mathrm{e}n}{5})^5\frac{31^{n-5}}{2^{5n-15}}<(\frac{\mathrm{e}n}{5})^5\,(\frac{31}{32})^{n-5}$, converges to 0 as $n\to\infty$. Therefore, showing $\sum_{6\le r\le b_n}\sigma'_{r,n}\xrightarrow{n\to\infty}0$ is sufficient for showing $\Phi'_n\xrightarrow{n\to\infty}0$. Because of $\frac{\mathrm{e}}{r}<\frac{1}{2}$, for every $6\le r\le b_n$ (this being the reason for analysing the first five summands separately) we have $\sigma'_{r,n}<(\frac{1}{2})^r\,n^r\,\frac{(2^r-1)^{n-r}}{2^{rn-\frac{1}{2}r^2-\frac{1}{2}r}}=n^r\,\frac{(2^r-1)^{n-r}}{2^{rn-\frac{1}{2}r^2+\frac{1}{2}r}}$. Therefore, showing $\Phi''_n:=\sum_{6\le r\le b_n}\sigma''_{r,n}\xrightarrow{n\to\infty}0$, where we have set $\sigma''_{r,n}:=n^r\,\frac{(2^r-1)^{n-r}}{2^{rn-\frac{1}{2}r^2+\frac{1}{2}r}}$, is sufficient for showing $\sum_{6\le r\le b_n}\sigma'_{r,n}\xrightarrow{n\to\infty}0$. Because of $\frac{(2^r-1)^{n-r}}{2^{rn-\frac{1}{2}r^2+\frac{1}{2}r}}<\frac{(2^r-1)^{n-r}}{2^{rn-r^2}}=(1-(\frac{1}{2})^r)^{n-r}$ for every $6\le r\le b_n$, proving that $\Phi'''_n:=\sum_{6\le r\le b_n}\sigma'''_{r,n}\xrightarrow{n\to\infty}0$, where $\sigma'''_{r,n}:=n^r\,(1-\frac{1}{2^r})^{n-r}$, is sufficient for showing that $\Phi''_n\xrightarrow{n\to\infty}0$. Since $\sigma'''_{r,n}/\sigma'''_{r+1,n}=n^{-1}\cdot(1-\frac{1}{2^{r+1}})\cdot(1-\frac{1}{2^{r+1}-1})^{n-r}<1$ for every $1\le r\le b_n$, it follows that $\Phi'''_n\le(b_n-6+1)\cdot\sigma'''_{b_n,n}\le b_n\cdot\sigma'''_{b_n,n}=\lfloor\frac{1}{2}\log n\rfloor\cdot n^{\lfloor\frac{1}{2}\log n\rfloor}\cdot\left(1-\frac{1}{2^{\lfloor\frac{1}{2}\log n\rfloor}}\right)^{n-\lfloor\frac{1}{2}\log n\rfloor}\le\frac{1}{2}\log n\cdot n^{\frac{1}{2}\log n}\cdot\left(1-\frac{1}{2^{\frac{1}{2}\log n}}\right)^{n-\frac{1}{2}\log n}\le[$ since $\frac{1}{2}\log n\le n^{\frac{1}{2}\log n}$ and $2^{\frac{1}{2}\log n}=n^{\log(\sqrt{2})}$ $]\le n^{\log n}\cdot\left(1+\frac{-1}{n^{\log(\sqrt{2})}}\right)^{n-\frac{1}{2}\log n}=:h(n)$, so we are done if we can prove $h(n)\xrightarrow{n\to\infty}0$. We have $\log(h(n))=(\log n)^2+(n-\frac{1}{2}\log(n))\cdot\log(1-\frac{1}{n^{\log(\sqrt{2})}})<(\log n)^2+n\cdot\log(1-\frac{1}{n^{\log(\sqrt{2})}})=:h_2(n)$. Notice that $h_2(n)\xrightarrow{n\to\infty}-\infty$ implies $\log(h(n))\xrightarrow{n\to\infty}-\infty$ and this implies $h(n)\xrightarrow{n\to\infty}0$. Therefore, we are done if we can prove $h_2(n)\xrightarrow{n\to\infty}-\infty$.

For every $-1<x<+1$ we have $\log(1+x)=\sum_{j\ge1}\frac{(-1)^{j-1}}{j}\,x^j$. Moreover, $-1<\frac{-1}{n^{\log(\sqrt{2})}}<0<+1$ for every $n\in\mathbb{N}$, so this expansion is applicable to our situation. We get $h_2(n)=(\log n)^2+n\cdot\sum_{j\ge1}\frac{(-1)^{j-1}}{j}\,(\frac{-1}{n^{\log(\sqrt{2})}})^j=(\log n)^2-n\cdot\sum_{j\ge1}\frac{1}{j}\cdot n^{-j\,\log(\sqrt{2})}=(\log n)^2-\sum_{j\ge1}\frac{1}{j}\cdot n^{1-j\,\log(\sqrt{2})}=(\log n)^2-n^{1-\log(\sqrt{2})}-\sum_{j\ge2}\frac{1}{j}\cdot n^{1-j\,\log(\sqrt{2})}<(\log n)^2-n^{1-\log(\sqrt{2})}$. Since $1-\log(\sqrt{2})>0$, we have $(\log n)^2-n^{1-\log(\sqrt{2})}\xrightarrow{n\to\infty}-\infty$, therefore $h_2(n)\xrightarrow{n\to\infty}-\infty$, and the proof is complete.

Problem 2.3 *Plane trees and binary trees.*
Find a bijection between plane trees with $n+1$ vertices and binary trees with n internal nodes.

A solution. Let $\mathcal{B}_k$ denote the set of all binary trees with k internal nodes. Let $\mathcal{P}_k$ denote the set of all plane trees with k nodes. In a binary tree at every internal node we may speak of a 'left descendant' and a 'right descendant', which we will abbreviate l and r . A bijection $\mathcal{B}_k \to \mathcal{P}_{k+1}$ can be be defined thus: given a $B \in \mathcal{B}_k$, draw B by—starting with the root of B—drawing the l up and the r to the right, and then recursively draw the subtrees at l and r in the same way.

For every $v \in B$, call the number of times one has to go left from the root to reach it the 'l-level of v'. Analogously define the 'r-level of v'. (With this definitions, the standard 'height' of a node in a tree is the sum of its l-level and its r-level.)

Now add a new node v_0 below the root of B and connect it to every *internal* node of B with l-level 0. (In the drawing, these are vertices immediately above v_0.) Now do the following for every $0 \leq i \leq m-2$ where m is the maximum l-level in the drawing of B: for every *internal* node v' in B having l-level i, connect it to every *internal* node v'' in B having the two properties that

(1) v'' lies in the subtree of B at v' ,

(2) l-level (v'') = l-level (v') + 1 .

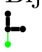
This defines a plane tree T such that

(1) T has one node more than B has internal nodes ,

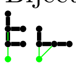
(2) the linear order on each subset of nodes of height h in T is given by the r-level of each such node in B .

This map is a bijection of the desired kind. It is visualized below (up to the case of 4 internal nodes; the green node at the bottom is the root of the respective plane tree).

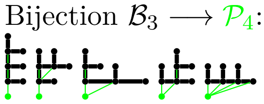
Bijection $\mathcal{B}_1 \longrightarrow \mathcal{P}_2$:



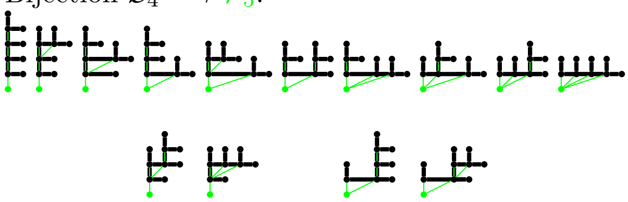
Bijection $\mathcal{B}_2 \longrightarrow \mathcal{P}_3$:



Bijection $\mathcal{B}_3 \longrightarrow \mathcal{P}_4$:



Bijection $\mathcal{B}_4 \longrightarrow \mathcal{P}_5$:



Problem 2.4 *Streaks of luck and enumeration.*
Given a fixed integer $k \geq 2$, let a_n be the number of binary words of length n not containing k consecutive zeroes. Compute the generating function $\sum_{n \geq 0} a_n z^n$.

A solution. Abbreviate $f := \sum_{n \geq 0} a_n z^n$. We will employ the symbolic method introduced in the lecture of 9 may 2012. The result will be an expression for the power series f as a product (of the form 'polynomial times inverse of another polynomial') of two elements of the rational function field

$\mathbb{C}(z) = \mathrm{Quot}(\mathbb{C}[z])$. We will be content with this answer, and not expand it into the power series form.

The following can easily be proved e.g. via an induction on the total number of ones; for the induction step, consider an arbitrary word w with m ones and remove the suffix starting at the last occurrence of a 1 in w:

Lemma 2 *Let 0^j denote the j-fold concatenation of 0 (in particular $0^0 := \epsilon$, the empty word). Every $w \in \{0,1\}^n$ not containing k consecutive zeroes can be decomposed as $w = 0^{j_0}(10^{j_1})(10^{j_2})\cdots(10^{j_t})$ with some integer $t \geq 0$ and $0 \leq j_\iota \leq k-1$ for every $0 \leq \iota \leq t$.* $\square$

Denote by $\mathcal{W}_k$ the combinatorial class of binary words which do not have any k consecutive zeroes. Denote by W_k its ordinary generating function.

We define the combinatorial class $\mathrm{SEQ}_{\leq k-1}(\{x\})$ to have underlying set $\bigcup_{0\leq \ell \leq k-1} \times^\ell \{x\}$ and word-length as the size function $|\cdot|$. Therefore its ordinary generating function (ogf) is the polynomial $\sigma_{k-1} := \sum_{0\leq\ell\leq k-1} z^\ell = \frac{1-z^k}{1-z}$. Moreover, we define $\mathrm{SEQ}(\{x\}) := \bigcup_{\ell\geq 0} \times^\ell\{x\}$. Its ogf is $\frac{1}{1-z}$.

From Lemma 2 we have

$$\mathcal{W}_k = \mathrm{SEQ}_{\leq k-1}(\{0\}) \times \mathrm{SEQ}(\{1\} \times \mathrm{SEQ}_{\leq k-1}(\{0\})) \quad . \tag{8}$$

This is an equality of sets to which we can apply (in this order) Theorem 1.6.2, then Theorem 1.6.3, and then again Theorem 1.6.2 from the supplementary material for the lecture of 9 may 2012, to get $W_k = \mathrm{ogf}(\mathcal{W}_k) = \sigma_{k-1} \cdot \frac{1}{1-z\cdot\sigma_{k-1}} = \frac{1-z^k}{1-z} \cdot \frac{1}{1-z\cdot\frac{1-z^k}{1-z}} = \frac{-z^k+1}{z^{k+1}-2z+1} \in \mathbb{C}(z)$, which is the kind of answer we were looking for.

Alternative solution. As seen above, the present problem does not need a recursive equation for the class we want to count. Nevertheless one *can* use a recursive equation to solve the problem: the following alternative road to the generating function $\frac{-z^k+1}{z^{k+1}-2z+1}$ was found in the tutorial of 16 may 2012: we have

$$\begin{aligned}\mathcal{W}_k = \{\} + \{0\} + \{0\}\times\{0\} + \cdots + \underbrace{\{0\}\times\cdots\times\{0\}}_{k-1 \text{ times}} \\ + \{1\}\times\mathcal{W}_k + \{0\}\times\{1\}\times\mathcal{W}_k + \cdots + \underbrace{\{0\}\times\cdots\times\{0\}}_{k-1\text{ times}}\times\{1\}\times\mathcal{W}_k \quad .\end{aligned} \tag{9}$$

Therefore $W_k = (1 + z + \cdots + z^{k-1}) + zW_k + z^2W_k + \cdots z^kW_k$, hence $W_k = \frac{1-z^k}{1-z}\cdot(2 - \frac{1-z^{k+1}}{1-z})^{-1}$ which is equal to the $\frac{-z^k+1}{z^{k+1}-2z+1}$ we found above.

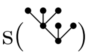
Problem 2.5 *Plane trees and ternary trees.*
Let T_n be the number of ternary trees with n internal nodes, and let E_n be the number of plane trees with n vertices such that every vertex has even outdegree. Find equations satisfied, respectively, by the generating functions

$$T(z) = \sum_{n\geq 0} T_n z^n \qquad \text{and} \qquad E(z) = \sum_{n\geq 0} E_n z^n \quad .$$

Deduce that $T_n = E_{2n+1}$ for all $n \geq 0$.

A solution.

We use the symbolic method introduced in the lecture of 9 may 2012.

Let $\mathcal{T}$ denote the countably-infinite set of all finite plane ternary trees. Define a size function $\mathrm{s}\colon \mathcal{T} \to \mathbb{Z}_{\geq 0}$ as the number of internal nodes. (So e.g. $\mathrm{s}(\bullet) = 0$, s() = 1 and s() = 2) The problem statement says that $T_n = [z^{\mathrm{s}=n}]\mathrm{ogf}(\mathcal{T})$.

Let $\mathcal{E}$ denote the countably-infinite set of all plane trees in which every vertex has even outdegree. Define a size function $\mathrm{s}\colon \mathcal{E} \to \mathbb{Z}_{\geq 0}$ as the total number of nodes. The problem statement says that $E_n = [z^{\mathrm{s}=n}]\mathrm{ogf}(\mathcal{E})$.

Writing + for disjoint union of sets, we have (as usual $\cdot^k$ means k-fold cartesian product):

(M.1) $\mathcal{T} = \{\emptyset\} + \{\bullet\} \times \mathcal{T}^3 \quad ,$

(M.2) $\mathcal{E} = \{\bullet\} \times (\{\emptyset\} + \mathcal{E}^2 + \mathcal{E}^4 + \cdots)$

Several times applying Theorem 1.6 from the supplementary material of the lecture on 9 may 2012 to the generating functions $T = \mathrm{ogf}(\mathcal{T}) \in \mathbb{C}[[z]]$ and $E = \mathrm{ogf}(\mathcal{E}) \in \mathbb{C}[[z]]$ we deduce from (M.1) and (M.2) the following equations in $\mathbb{C}[[z]]$:

(F.1) $T = 1 + z \cdot T^3 \quad ,$

(F.2) $E = z(1 + E^2 + E^4 + \cdots) = z \cdot (1 - E^2)^{-1} \qquad \Longleftrightarrow \qquad E^3 - E + z = 0 \quad .$

The equation $x^3 - x + z = 0$, which according to (F.2) is satisfied by E, has at most one solution x in the subset $\mathrm{N} := \{f \in \mathbb{C}[[z]] \colon [z^0](f) \geq 1\}$: for suppose there were $f, g \in \mathrm{N}$ with $f \neq g$ which are both solutions of $x^3 - x + z = 0$. Then $f - g = f^3 - g^3 =$ [as in any commutative ring] $(f - g) \cdot (f^2 + fg + g^2)$. On the one hand, since $f - g \neq 0$ by assumption, and since $\mathbb{C}[[z]]$ is an integral domain, it follows that $f^2 + fg + g^2 = 1$, hence $[z^0](f^2 + fg + g^2) = [z^0](1) = 1$. On the other hand, $[z^0](f^2 + fg + g^2) = [z^0](f^2) + [z^0](fg) + [z^0](g^2) =$ [by definition of multiplication of power series and by the assumpton $f, g \in \mathrm{N}$] $\geq 1 + 1 + 1 = 3$. Contradiction.

Now notice that $z \cdot T(z^2) =$ [by (F.1)] $= z \cdot (1 + z^2 \cdot (T(z^2))^3) = z + (z\, T(z^2))^3$, hence the power series $z \cdot T(z^2)$ satisfies the equation $x^3 - x + z = 0$ discussed in the preceding paragraph. By what was proved there, and since both $E \in \mathrm{N}$ and $T \in \mathrm{N}$, it follows that $E = z \cdot T(z^2)$. Hence $E_{2n+1} = [z^{2n+1}]E = [z^{2n+1}](z\, T(z^2)) = [z^n]T(z) = T_n$, which is what we had to prove.