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Sheet 3

Throughout this sheet, 'GF' means 'exponential generating function'.

Problem 3.1 *Enumeration of special graphs.*
Here, 'graph' means 'labelled graph'.

(1) Given that there are $2^{\binom{n}{2}}$ graphs with n vertices, find an expression for the GF of *connected* graphs. Use Maple to compute the first coefficients.

(2) Find the GF of 2-regular labelled graphs.

(3) A graph is *unicyclic* if it is connected and has a unique cycle. Find the GF of unicyclic graphs in terms of the GF $T(z)$ of rooted labelled trees. Enrich the previous GF by encoding the length of the unique cycle by means of a secondary variable.

Problem 3.2 *Counting mappings.*
In this problem we consider (arbitrary) mappings $f\colon [n] \to [n]$ where $[n] = \{1, 2, \ldots, n\}$. Represent a mapping f by a directed graph on the vertex set $[n]$ in which the directed edges are $(i, f(i))$ for $i \in [n]$. Such graphs are called functional graphs and are characterized by the fact that each vertex has outdegree 1.

(1) Show that a functional graph is a set of connected functional graphs, and that a connected functional graph is a collection of rooted trees arranged in a directed cycle of length at least one.

(2) Let $M(z)$ be the GF of mappings, where the size is the number of elements. Express $M(z)$ in terms of the GF $T(z)$ of rooted labelled trees.

(3) Compute the GF of mappings without fixed points. Compute the GF of mappings counted according to the number of cyclic points (those lying in the cycles of the graph).

Problem 3.3 *Enumeration of special permutations.*
A permutation is a bijection of a finite set to itself.

(1) Compute the GF of involutions, that is, permutations σ satisfying $\sigma^2 = 1$.

(2) More generally, for $r \geq 2$, compute the GF of permutations satisfying $\sigma^r = 1$.

(3) A permutation σ is called alternating if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$. Compute the GF of alternating permutations of odd length. Do the same for those of even length.

Problem 3.4 *A correspondence between permutations.*
A *record* in a permutation $\sigma_1, \ldots, \sigma_n$ is an element σ_j such that $\sigma_i < \sigma_j$ for all $i < j$. Consider the following correspondence between permutations. Given a permutation $\sigma = c_1 \cdots c_k$ decomposed as a product of disjoint cycles $c_1, \ldots, c_k$, write each cycle with the leader (maximum element) in the first

position, and order the cycles by increasing values of their leaders. Then erase parentheses to obtain a new permutation σ'. For example, given $\sigma = (2,5,7,9)\,(4,3)\,(6,8,1)$, rewrite it as

$$\sigma = (\mathbf{4},3)\,(\mathbf{8},1,6)\,(\mathbf{9},2,5,7) \quad , \tag{1}$$

so that

$$\sigma' = 4,3,8,1,6,9,2,5,7 \quad . \tag{2}$$

Show that the correspondence $\sigma \to \sigma'$ is a bijection. ~~, and that the number of records of σ' is equal to the number of records of σ.~~ Deduce that the number of permutations of length n with k records is the Stirling number of the first kind $\left[{n \atop k}\right]$.

An erratum. The part "and that the number of records of σ' is equal to the number of records of σ." of the problem statement is both false and superfluous for the proof of the (correct) statement about the link to Stirling numbers. It is false since for example the permutation $\sigma := \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{smallmatrix}\right)$ has exactly one record (namely $\sigma_1 = 6$) but σ', computed from $\sigma = (\mathbf{4},3)(\mathbf{5},2)(\mathbf{6},1)$, is $\sigma' = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 2 & 6 & 1 \end{smallmatrix}\right)$ and has exactly 3 records (namely $\sigma_1 = 4$, $\sigma_3 = 5$ and $\sigma_5 = 6$). Considering larger examples of this type one sees that the difference of the number of records of σ and σ' can become arbitrarily large: for even n, the permutation σ defined by $\sigma_i := n - i + 1$ has exactly one record, but σ' then has exactly $\frac{n}{2}$ records.