Random Graphs (MA 5208) Summer 2012

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Sheet 4

Problem 4.1 *Composing the binomial distribution with a uniform distribution.*
Let X have the binomial distribution $\mathrm{bin}(n, U)$, where U is uniform on $(0, 1)$. Show that X is uniformly distributed on $\{0, 1, \ldots, n\}$.

Problem 4.2 *Generating function for size of the n-th generation.*
Let the offspring distribution of a branching process be $p_k = (1-p)p^k$ for $k \geq 0$ and $0 \leq p \leq 1$. Show that the generating function $G_n(s)$ of the family size Z_n at generation n (assuming $Z_0 = 1$) is equal to

$$G_n(u) = \begin{cases} \frac{n-(n-1)u}{n+1-nu} & \text{if } p = q = 1/2 \quad , \\ \frac{q[p^n - q^n - pu(p^{n-1} - q^{n-1})]}{p^{n+1} - q^{n+1} - pu(p^n - q^n)} & \text{if } p \neq q \quad . \end{cases} \tag{1}$$

Deduce that the probability of ultimate extinction is

$$\lim_{n\to\infty} \mathsf{Pr}[Z_n = 0] = \begin{cases} 1 & \text{if } p \leq q \quad , \\ q/p & \text{if } q < p \quad . \end{cases} \tag{2}$$

Problem 4.3 *Distribution of the extinction time.*
In the same situation as in the previous problem, let $T = \min\{n \colon Z_n = 0\}$ be the random variable equal to the extinction time. Find $\mathsf{Pr}[T = n]$. For what values of p do we have $\mathsf{Ex}[T] < +\infty$?

Problem 4.4 *How sizes of generations correlate with each other.*
Let Z_n be the size of the n-th generation in a branching process with $Z_0 = 1$, $\mathsf{Ex}[Z_1] = \mu$ and $\sigma^2(Z_1) > 0$. Compute $\mathsf{Ex}[Z_n|Z_m]$ and show that $\mathsf{Ex}[Z_n Z_m] = \mu^{n-m}\,\mathsf{Ex}[Z_m^2]$ for $m \leq n$. Compute $\rho(Z_m, Z_n)$, where $\rho(\cdot,\cdot)$ denotes the standard Pearson correlation coefficient. [Hint: distinguish whether $\mu = 1$ or not.]