Prof. Dr. Marc Noy | Prof. Dr. Anusch Taraz | M.Sc. Peter Heinig

Sheet 5

Problem 5.1 *Random regular graphs: the inner workings of the configuration model.*
Let G be a d-regular multigraph with vertex set $[n]$. Prove a formula for (w.r.t. the configuration model from the lecture of 13 june 2012) the number of configurations giving rise to G.

Problem 5.2 *Random regular graphs: number of copies of a graph w.r.t. to the configuration model.*
Let $h \in \mathbb{Z}_{\geq 1}$, let H be a (fixed) finite simple graph on $[h]$, i.e. $H = ([h], E)$ with $E \subset \binom{[h]}{2}$. Let $\mathrm{G}^*(n, d)$ denote the probability space over d-regular multigraphs on $[n]$ defined in the lecture of 13 june 2012. Prove that $\mathsf{Ex}_{\mathrm{G}^*(n,d)}[\text{number of copies of } H \text{ in } G^*(n, d)] \in O(n^{|H|-\|H\|})$.

Problem 5.3 *An infinite expected number, yet no chance of having any.*
Let X give the number of spanning trees of a graph. Find a function $p_n \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that w.r.t. $\mathrm{G}(n, p_n)$ we have $\mathsf{Ex}[X] \xrightarrow{n\to\infty} \infty$ and $\mathsf{Pr}[X \geq 1] \xrightarrow{n\to\infty} 0$.

Problem 5.4 *The threshold for the existence of a copy of an arbitrary fixed graph.*
Formulate and prove a statement which generalizes Theorem 3.4 of the lecture of 20 june 2012 from strictly balanced to arbitrary graphs H.

Problem 5.5 *Branching processes as a tool in analysing Erdős–Rényi random graphs.*
For every $\ell \in \mathbb{Z}_{\geq 0}$ let $\mathrm{Z}_\ell^{(n,p_n)}$ denote the size of the ℓ-th generation in the basic Galton–Watson branching process with $\mathrm{Z}_0 = 1$ and the offspring given by the binomial distribution $\mathrm{bin}(n, p_n)$. Let $\mathrm{org}_{n,p_n} := \sum_{\ell\geq 0} \mathrm{Z}_\ell^{n,p_n}$ denote the random variable giving the number of all organisms who ever lived in the process (org_{n,p_n} has image $\mathbb{Z}_{\geq 1} \cup \{\infty\}$ and on a given process ω in the sample space Ω evaluates to either ∞ or an element of $\mathbb{Z}_{\geq 1}$, according to whether ω contains an infinite path). If $G = ([n], E)$ is a graph on $[n]$, and if $i \in [n]$, we denote by $G(i)$ the connected component of G containing i. Let $\mathsf{Pr}_{\mathrm{G}(n,p_n)}$ denote the measure of $\mathrm{G}(n, p_n)$ and let $\mathsf{Pr}_{\mathrm{GW}(n,p_n)}$ denote the measure of the branching process described above. Prove that for every fixed $s \in \mathbb{Z}_{\geq 1}$,

$$\mathsf{Pr}_{\mathrm{G}(n,p_n)}[\{G = ([n], E) \colon |G(1)| \geq s\}] \leq \mathsf{Pr}_{\mathrm{GW}(n,p_n)}[\{\omega \in \Omega \colon \mathrm{org}_{n,p_n}(\omega) \geq s\}] \quad . \tag{1}$$

Hint: Define and analyse a stochastic process which starts at vertex 1 and 'uncovers' the random graph in a breadth-first manner.