Technische Universität München

Zentrum Mathematik

Random Graphs (MA 5208) Summer 2012

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Sheet 5

Problem 5.1 *Random regular graphs: the inner workings of the configuration model.*
Let G be a d-regular multigraph with vertex set $[n]$. Prove a formula for (w.r.t. the configuration model from the lecture of 13 june 2012) the number of configurations giving rise to G.

A solution. Let G be a multigraph with vertex-set $[n]$. For every $i \in [n]$ let the *loop-degree* $\mathrm{ld}_G(i)$ of i denote the number of loops of G at i. (In particular $\mathrm{ld}_G(i) = 0$ if and only if there is no loop at i.) For every $\{i_1, i_2\} \in \binom{[n]}{2}$ let the *multiple-edge-degreee* $\mathrm{md}_G(i_1, i_2)$ denote the multiplicity of the edge of G between i_1 and i_2. (In particular, $\mathrm{md}_G(i_1, i_2) = 0$ if and only if i_1 and i_2 are not connected in G.)

One can prove that the solution to Problem 5.1 is the following formula. A proof can be found in the supplement published alongside this solution sheet. We declare that supplement to be non-examinable w.r.t. the final exam. However, the map π and the formula itself are part of the potentially examinable material.

Proposition 1 (number of configurations giving rise to a given multigraph)**.** *For every even $nd \geq 2$, every d-regular multigraph G with vertex set $[n]$, and with π denoting the map from pairings of $[n]\times[d]$ to such multigraphs which was defined in the lecture of 13 june 2012,*

$$|\pi^{-1}(G)| = \frac{(d!)^n}{\left(\prod_{i\in[n]}(2^{\mathrm{ld}_G(i)}\cdot \mathrm{ld}_G(i)!)\right)\cdot\left(\prod_{\{i_1,i_2\}\in\binom{[n]}{2}} \mathrm{md}_G(i_1,i_2)!\right)} \quad . \tag{1}$$

An example for formula (1).

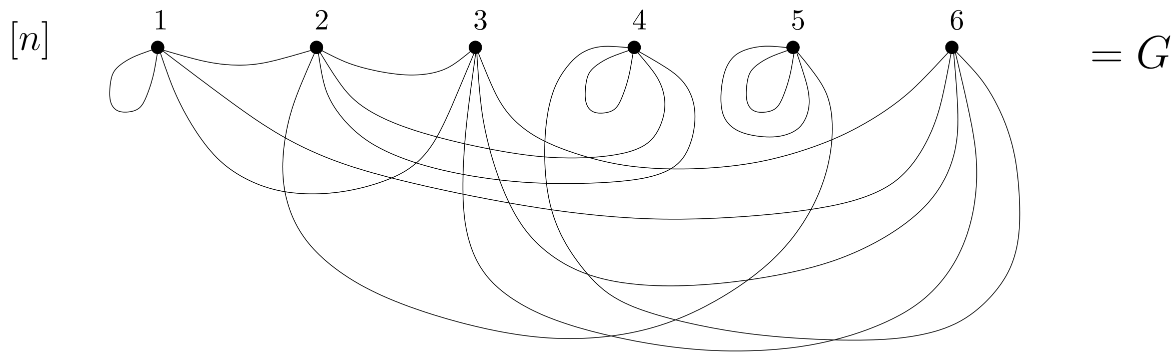

Figure 1: An example with $n = 6$ and $d = 5$: this visualizes a 5-regular multigraph on 6 vertices. By (1), we have $|\pi^{-1}(G)| = \frac{(5!)^6}{(1!\cdot 2^1 \ \cdot\ 1!\cdot 2^1 \ \cdot\ 2!\cdot 2^2)\cdot(2!\cdot 3!)} = 7776000000$.

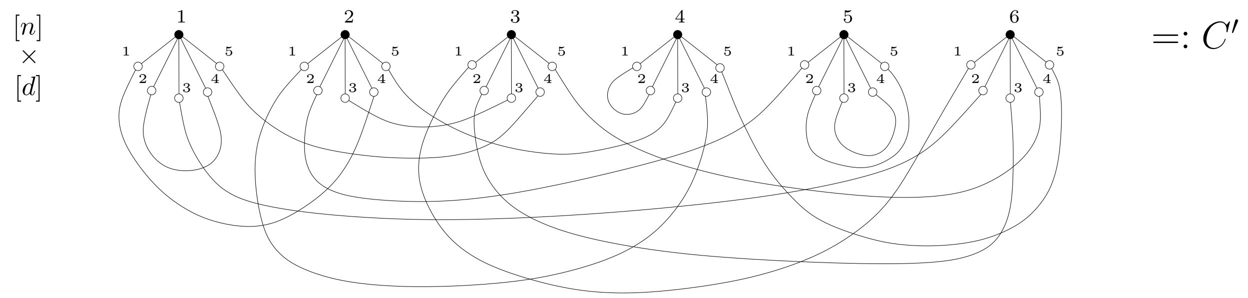

Figure 2: This visualizes a configuration C' in the preimage of the G from Figure 1 w.r.t. π. The stabilizer subgroup of C' in $\times^6 \mathfrak{S}_{[5]}$ w.r.t. the action α defined in the supplement to Problem 5.1 has exactly $\bigl(1! \cdot 2^1 \;\cdot\; 1! \cdot 2^1 \;\cdot\; 2! \cdot 2^2\bigr) \cdot \bigl(2! \cdot 3!\bigr) = 384$ elements. (First bracketed factor due to loops, second bracketed factor due to multiple edges.) Therefore, by (1), we have $|\pi^{-1}(G)| = \frac{(5!)^6}{384} = 7776000000$.

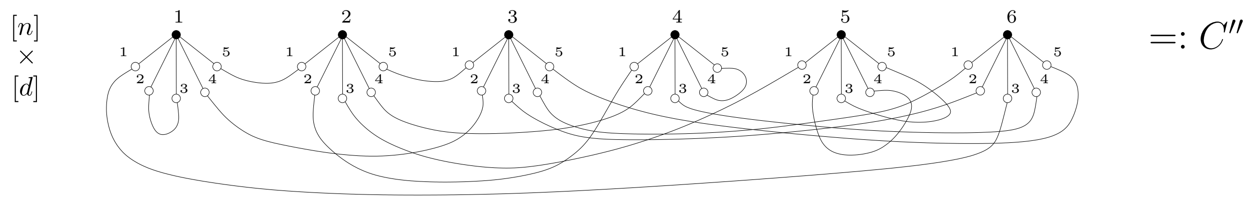

Figure 3: This visualizes another configuration C'' in the preimage of the G from Figure 1 w.r.t. π. The stabilizer subgroup of C'' in $\times^6 \mathfrak{S}_{[5]}$ w.r.t. the action α defined in the supplement to Problem 5.1 has, just as for C', exactly $\bigl(1! \cdot 2^1 \;\cdot\; 1! \cdot 2^1 \;\cdot\; 2! \cdot 2^2\bigr) \cdot \bigl(2! \cdot 3!\bigr) = 384$ elements.

Problem 5.2 *Random regular graphs: number of copies of a graph w.r.t. to the configuration model.* Let $h \in \mathbb{Z}_{\geq 1}$, let H be a (fixed) finite simple graph on $[h]$, i.e. $\mathrm{H} = ([h], E)$ with $E \subset \binom{[h]}{2}$. Let $\mathrm{G}^*(n,d)$ denote the probability space over d-regular multigraphs on $[n]$ defined in the lecture of 13 june 2012. Prove that $\mathsf{Ex}_{\mathrm{G}^*(n,d)}[\text{number of copies of H in } \mathrm{G}^*(n,d)] \in O(n^{|\mathrm{H}|-\|\mathrm{H}\|})$.

A solution. A solution will be published somewhere in the future. We declare Problem 5.2 to be non-examinable w.r.t. the final exam.

Remark. The topic of random regular graphs made only a brief appearance in our lecture, so we would like to emphasize its applicability both within and without mathematics: it is easy to image situations where you need (e.g. for experiments or the allocation of tasks) a graph without any apparent structure or 'bias', and at the same time any irregularity in the degrees would immediately be noticed and has to be avoided (ruling out the use of $\mathrm{G}(n,p)$ with its small fluctuations in the vertex-degrees). The probability space $\mathrm{G}^*(n,d)$ provides a means for generating such graphs and Problem 5.1 and Problem 5.2 give you a start on how to quantitatively control this model (in particular to quantify how unlikely it is that loops or multiple edges occur: as mentioned in the lecture of 13 june 2012 the chance of a random element of $\mathrm{Conf}_{[n]\times[d]}$ mapping to a simple d-regular graph is approximately $\exp(-\frac{1}{4}(d^2-1))$ for large n).

Problem 5.3 *An infinite expected number, yet no chance of having any.*
Let X give the number of spanning trees of a graph. Find a function $p_n\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that w.r.t. $\mathrm{G}(n,p_n)$ we have $\mathsf{Ex}[X] \xrightarrow{n\to\infty} \infty$ and $\mathsf{Pr}[X \geq 1] \xrightarrow{n\to\infty} 0$.

In this solution, Pr and Ex refer to the probability space $\mathrm{G}(n,p_n)$. Let $\mathcal{G}_n = \bigcup_{\emptyset \subset F \subset \binom{[n]}{2}} \{([n],F)\}$ denote the set of all graphs with vertex set $[n]$. Let $\mathcal{T}$ denote the set of all n^{n-2} spanning trees of $\binom{[n]}{2}$. For every $T \in \mathcal{T}$ let $\mathbf{1}_T\colon \mathcal{G}_n \to \{0,1\}$ be defined by $\mathbf{1}_T(G) := 1$ if and only if $\mathrm{E}(T) \subset \mathrm{E}(G)$. Then $X = \sum_{T\in\mathcal{T}}$, hence $\mathsf{Ex}[X] = \sum_{T\in\mathcal{T}} \mathsf{Pr}[\mathrm{E}(T) \subset \mathrm{E}(G)] = n^{n-2} \cdot p_n^{n-1}$.

Note that $\{\ G \in \mathcal{G}_n\colon\ X(G) \geq 1\ \} = \{\ G \in \mathcal{G}_n\colon\ \}$. You know from the lecture of 20 june 2012 that if $p_n/((\log n)/n) \xrightarrow{n\to\infty} 0$, then $\mathsf{Pr}\ [\ \{\ G \in \mathcal{G}_n\colon\ X(G) \geq 1\ \}\ ] \xrightarrow{n\to\infty} 0$. With $p_n := \frac{1+\varepsilon}{n}$ for an arbitrary fixed $\varepsilon > 0$, this is the case, and at the same time we have $\mathsf{Ex}[X] = n^{n-2}p_n^{n-1} = \frac{(1+\varepsilon)^{n-1}}{n} \xrightarrow{n\to\infty} \infty$, so we have found a function of the required kind, completing the solution of Problem 5.3.

Remark. This is a good place for a summary of the asymptotically-almost-sure behaviour of $\mathrm{G}(n,p_n)$ at $p_n = \frac{1+\varepsilon}{n}$: there exist absolute constants $\varepsilon_0 > 0$ and $c > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $p_n = \frac{1+\varepsilon}{n}$, then w.r.t. the probability space $\mathrm{G}(n,p_n)$ we simultaneously have

(1) a diverging expected number of spanning trees ,

(2) asymptotically almost surely a non-connected graph ,

(3) asymptotically almost surely a path of length $c \cdot \varepsilon^2 \cdot n$,

(4) asymptotically almost surely no path of length $c \cdot \varepsilon \cdot n$,

(5) asymptotically almost surely a connected component with $c \cdot \varepsilon \cdot n$ vertices .

Problem 5.4 *The threshold for the existence of a copy of an arbitrary fixed graph.*
Formulate and prove a statement which generalizes Theorem 3.4 of the lecture of 20 june 2012 from strictly balanced to arbitrary graphs H.

A solution. We recall that if H' and H are graphs then $H' \subset H$ if and only if $\mathrm{V}(H') \subset \mathrm{V}(H)$ and $\mathrm{E}(H') \subset \mathrm{E}(H)$.

If $[h] \subset [n]$, H is a graph on $[h]$ and G a graph on $[n]$, then we define

$$H \hookrightarrow G \qquad :\Longleftrightarrow \qquad \text{there exists an injective graph homomorphism } H \to G \quad . \tag{2}$$

For the desired statement, it is enough to replace $\rho(H)$ in Theorem 3.4 of the lecture of 20 june 2012 by $\mathrm{mld}(H) := \max \frac{\|H\|}{|H|}$, where mld is for 'maximum local density' and the maximum is computed over all subgraphs of H with at least one vertex:

Theorem 2 *For every fixed graph H,*

$$\mathsf{Pr}_{p_n}[\{G \in \mathcal{G}_n\colon H \hookrightarrow G\}] \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } p_n \ll n^{-\frac{1}{\mathrm{mld}(H)}} \quad , \\ 1 & \text{if } p_n \gg n^{-\frac{1}{\mathrm{mld}(H)}} \quad . \end{cases} \tag{3}$$

For a proof of Theorem 2, it is enough to notice that since $\mathrm{mld}(H)$ is a maximum over a finite set, there exists at least one graph H_0 with $\frac{\|H_0\|}{|H_0|} = \mathrm{mld}(H)$. Then we have $n^{-\frac{1}{\mathrm{mld}(H)}} = n^{-\frac{|H_0|}{\|H_0\|}}$. Now it only remains to convince oneself that one may repeat the proof which for balanced H was given in the lecture of 20 june 2012.

Remark. It can be proved that every graph H is contained in a *balanced* (in the sense of the lecture of 20 june 2012) graph $\widetilde{H} \supseteq H$. This obviously implies Theorem 2. However, while the literature contains a quite elementary proof for this, this proof is rather complicated and to this day there does not seem to be known a simple argument for the existence of such a balanced extension. Maybe one of you will once find a short conceptual proof for this?

Problem 5.5 *Branching processes as a tool in analysing Erdős–Rényi random graphs.*
For every $\ell \in \mathbb{Z}_{\geq 0}$ let $\mathrm{Z}_\ell^{(n,p_n)}$ denote the size of the ℓ-th generation in the basic Galton–Watson branching process with $\mathrm{Z}_0 = 1$ and the offspring given by the binomial distribution $\mathrm{bin}(n, p_n)$. Let $\mathrm{org}_{n,p_n} := \sum_{\ell \geq 0} \mathrm{Z}_\ell^{n,p_n}$ denote the random variable giving the number of all organisms who ever lived in the process (org_{n,p_n} has image $\mathbb{Z}_{\geq 1} \cup \{\infty\}$ and on a given process ω in the sample space Ω evaluates to either ∞ or an element of $\mathbb{Z}_{\geq 1}$, according to whether ω contains an infinite path). If $G = ([n], E)$ is a graph on $[n]$, and if $i \in [n]$, we denote by $G(i)$ the connected component of G containing i. Let $\mathsf{Pr}_{\mathrm{G}(n,p_n)}$ denote the measure of $\mathrm{G}(n, p_n)$ and let $\mathsf{Pr}_{\mathrm{GW}(n,p_n)}$ denote the measure of the branching process described above. Prove that for every fixed $s \in \mathbb{Z}_{\geq 1}$,

$$\mathsf{Pr}_{\mathrm{G}(n,p_n)}[\{G = ([n], E) \colon |G(1)| \geq s\}] \leq \mathsf{Pr}_{\mathrm{GW}(n,p_n)}[\{\omega \in \Omega \colon \mathrm{org}_{n,p_n}(\omega) \geq s\}] \quad . \tag{4}$$

Hint: Define and analyse a stochastic process which starts at vertex 1 and 'uncovers' the random graph in a breadth-first manner.

A solution. A solution will be published somewhere in the future. We declare Problem 5.5 to be non-examinable w.r.t the final exam.