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Zentrum Mathematik
Random Graphs (MA 5208) Summer 2012
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Supplement: A proof for the formula asked in Problem 5.1

Concerning the fact that this solution came out a bit lengthy: should the final answer in Problem 5.1 be obvious to you, fine. But then you are still left with the exposition problem of how to convince someone else who is not as clairvoyant. This solution offers you one structured way of proving the formula within the context of the basic theory of group actions. Besides, we do not know of any published full proof of the formula (19) (only some brief, sometimes misleading sketches), so we think that with this solution we add something new to the literature. At the same time, we did not write out every detail and leave that to you on two occasions towards the end. Probably we will complete the solution in the future.

We start by recalling and extending a few things mentioned in the lecture of 13 june 2012:

Definition 1 (set of all possible configurations)**.** *Let $dn \geq 2$ be even. We define* $\mathrm{Conf}_{[n]\times[d]}$ *as the set of all partitions of the set* $[d] \times [n]$ *into* 2*-element sets.*

Definition 2 (multigraph)**.** *Syntactically, we define a finite multigraph G as a triple (V, E, g) with V and E finite disjoint sets and $\mathrm{g}\colon E \to V \dot\cup \binom{V}{2}$ an arbitrary map (sometimes referred to as* attaching map*). The intended meaning*[1] *is that $\mathrm{g}(e)$ tells us the set of endpoints of $e \in E$.*

Definition 3 (set of d-regular multigraphs on $[n]$)**.** *For every even $nd \geq 2$ we denote by $\mathrm{E}_{n,d} = \{e_1^{n,d}, \ldots, e_{\frac{nd}{2}}^{n,d}\}$ some fixed $\frac{nd}{2}$-element set and define*

$$\mathrm{Mult}_{n,d} := \bigcup_{\mathrm{g}} \{([n], \mathrm{E}_{n,d}, \mathrm{g})\} \quad , \tag{1}$$

where the union is over all maps $g\colon \mathrm{E}_{n,d} \to [n] \dot\cup \binom{[n]}{2}$ having the property that for every $i \in [n]$

$$|\{e \in E \colon i \in \mathrm{g}(e) \in \binom{[n]}{2}\}| + 2\,|\{e \in \mathrm{E}_{n,d} \colon i = \mathrm{g}(e)\}| = d \quad . \tag{2}$$

Definition 4 (loop, ordinary edge, part of a multi-loop, part of a multi-edge, end of a multi-loop, ends of a multi-edge)**.** *If $G = (V, E, \mathrm{g})$ is a multigraph as in Definition 2, an $e \in \mathrm{E}_{n,d}$ is called*

(1) loop *if and only if* $\mathrm{g}(e) \in V$,

(2) ordinary edge *if and only if* $\mathrm{g}(e) \in \binom{V}{2}$,

(3) part of a multi-loop *if and only if it is a loop and* $|\mathrm{g}^{-1}(\mathrm{g}(e))| \geq 2$,

(4) part of a multi-edge *if and only if it is an ordinary edge and* $|\mathrm{g}^{-1}(\mathrm{g}(e))| \geq 2$.

[1] Exercise: say in your own words what it means (1) when $\mathrm{im}(\mathrm{g}) \subsetneq V$, (2) when $\mathrm{im}(\mathrm{g}) = V$ and (3) when s is a constant map.

A vertex $v \in V$ *is called* end of a multi-loop *if and only if* $|\mathrm{g}^{-1}(v)| \geq 2$.

A 2-set $S \in \binom{V}{2}$ *is called* the ends of a multi-edge *if and only if* $|\mathrm{g}^{-1}(S)| \geq 2$.

Definition 5 (loop-degree of an $i \in [n]$ within a multigraph, multi-edge-degree of an $\{i_1, i_2\} \in \binom{[n]}{2}$ within a multigraph)**.** *For every even* $nd \geq 2$*, every multigraph* $G = ([n], E, \mathrm{g})$ *we define*

(1) *for every* $i \in [n]$ *the* loop-degree of i in G *as* $\mathrm{ld}_G(i) := |\{e \in E \colon \mathrm{g}(e) = i\}|$,

(2) *for every* $\{i_1, i_2\} \in \binom{[n]}{2}$ *the* multi-edge-degree of $\{i_1, i_2\} \in \binom{[n]}{2}$ in G *as* $\mathrm{ld}_G(i_1, i_2) := |\{e \in E \colon \mathrm{g}(e) = \{i_1, i_2\}\}|$.

We now formalize the map π mentioned in the lecture of 13 june 2012:

Definition 6 *For even* $nd \geq 2$ *we define a map*

$$\pi \colon \mathrm{Conf}_{[n]\times[d]} \longrightarrow \mathrm{Mult}_{n,d} \tag{3}$$

as follows: let an arbitrary $C \in \mathrm{Conf}_{[n]\times[d]}$ *be given. Let* $\preceq$ *denote the lexicographic order on* $[n] \times [d]$*. Since* $\preceq$ *is a linear order, there is exactly one way of writing*

$$\begin{aligned} C = \Big\{ & \{(i_{1,1}, j_{1,1}), (i_{2,1}, j_{2,1})\}, \\ & \{(i_{1,2}, j_{1,2}), (i_{2,2}, j_{2,2})\}, \\ & \{(i_{1,3}, j_{1,3}), (i_{2,3}, j_{2,3})\}, \\ & \ldots, \\ & \{(i_{1,\frac{nd}{2}}, j_{1,\frac{nd}{2}}), (i_{2,\frac{nd}{2}}, j_{2,\frac{nd}{2}})\} \Big\} \end{aligned} \tag{4}$$

such that

(Ord.1) $(i_{1,\kappa}, j_{1,\kappa}) \preceq (i_{2,\kappa}, j_{2,\kappa})$ *for every* $1 \leq \kappa \leq \frac{nd}{2}$,

(Ord.2) $(i_{1,\kappa}, i_{2,\kappa}) \preceq (i_{1,\kappa+1}, i_{2,\kappa+1})$ *for every* $1 \leq \kappa \leq \frac{nd}{2} - 1$,

(Ord.3) $(i_{1,\kappa}, i_{2,\kappa}) = (i_{1,\kappa+1}, i_{2,\kappa+1})$ *implies* $(j_{1,\kappa}, j_{2,\kappa}) \preceq (j_{1,\kappa+1}, j_{2,\kappa+1})$*, for every* $1 \leq \kappa \leq \frac{nd}{2} - 1$.

We then define

$$\mathrm{g}^{\pi}_C(e^{n,d}_{\kappa}) := \{i_{1,\kappa}, i_{2,\kappa}\} \in [n] \,\dot{\cup}\, \binom{[n]}{2} \quad , \tag{5}$$

for every $1 \leq \kappa \leq \frac{nd}{2}$*. This defines an attaching map* $g^{\pi}_C \colon \mathrm{E}_{n,d} \to [n] \,\dot{\cup}\, \binom{[n]}{2}$*. Hence*

$$\pi(C) := ([n], \mathrm{E}_{n,d}, \mathrm{g}^{\pi}_C) \tag{6}$$

defines a map $\pi \colon \mathrm{Conf}_{[n]\times[d]} \longrightarrow \mathrm{Mult}_{n,d}$.

Remark 7 *The hypothesis in the implication in condition* (Ord.3) *is true if and only if*

$$\begin{aligned} & (i_{1,\kappa} = i_{2,\kappa} =: i \text{ is the end of a multi-loop}) \\ & \text{or} \\ & (i_{1,\kappa} \neq i_{2,\kappa} \text{ and } \{i_{1,\kappa}, i_{2,\kappa}\} = \{i_{1,\kappa+1}, i_{2,\kappa+1}\} \text{ are the ends of a multi-edge}). \end{aligned} \tag{7}$$

It is convenient to overload the symbol π in the following way:

Definition 8 *For any 2-set of pairs* $\{(a,b),(c,d)\}$ *we define* $\pi(\{(a,b),(c,d)\}) := \{a,c\}$.

Note that Definition 8 is—despite not demanding (a,b) to be in some sense smaller than (c,d)—unambiguous: the result is a 2-set. The justification for using the symbol π in Definition 8 is that if $C = \{\ \{(i_{1,1},j_{1,1}),\ (i_{2,1},j_{2,1})\},\ \{(i_{1,2},j_{1,2}),\ (i_{2,2},j_{2,2})\},\ \{(i_{1,3},j_{1,3}),\ (i_{2,3},j_{2,3})\},\ \dots,\ \{(i_{1,\frac{nd}{2}},j_{1,\frac{nd}{2}}),$ $(i_{2,\frac{nd}{2}},j_{2,\frac{nd}{2}})\}\ \}$ is an element of $\mathrm{Conf}_{[n]\times[d]}$ written so as to satisfy (Ord.1), (Ord.2) and (Ord.3), then for every $1 \le \kappa \le \frac{nd}{2}$,

$$\mathrm{g}_C^{\pi}(e_\kappa^{n,d}) = \pi(\{(i_{1,\kappa},j_{1,\kappa}),(i_{2,\kappa},j_{2,\kappa})\}) \quad . \tag{8}$$

If M is a set we denote by '$\mathfrak{S}_M$' or '$\mathfrak{S}(M)$' the group of all permutations of M. We denote by $\Gamma := \times^n\ \mathfrak{S}_{[d]}$ the n-fold direct product of the symmetric group on $[d]$.

Definition 9 (the group action α)**.** *We define the group homomorphism*

$$\alpha\colon \Gamma \longrightarrow \mathfrak{S}(\mathrm{Conf}_{[n]\times[d]}) \tag{9}$$

which maps each $\gamma = (\gamma_1,\dots,\gamma_n) \in \Gamma$ *to the bijection* $\mathrm{Conf}_{[n]\times[d]} \to \mathrm{Conf}_{[n]\times[d]}$ *which the partition*

$$\begin{aligned} C = \Big\{ & \{(i_{1,1},j_{1,1}),(i_{2,1},j_{2,1})\},\\ & \{(i_{1,2},j_{1,2}),(i_{2,2},j_{2,2})\},\\ & \{(i_{1,3},j_{1,3}),(i_{2,3},j_{2,3})\},\\ & \dots,\\ & \{(i_{1,\frac{nd}{2}},j_{1,\frac{nd}{2}}),(i_{2,\frac{nd}{2}},j_{2,\frac{nd}{2}})\}\Big\} \end{aligned} \tag{10}$$

of $[n]\times[d]$ *to the partition*

$$\begin{aligned} \alpha(\gamma)(C) := \Big\{ & \{(i_{1,1},\gamma_{i_{1,1}}(j_{1,1})),(i_{2,1},\gamma_{i_{2,1}}(j_{2,1}))\},\\ & \{(i_{1,2},\gamma_{i_{1,2}}(j_{1,2})),(i_{2,2},\gamma_{i_{2,2}}(j_{2,2}))\},\\ & \{(i_{1,3},\gamma_{i_{1,3}}(j_{1,3})),(i_{2,3},\gamma_{i_{2,3}}(j_{2,3}))\},\\ & \dots,\\ & \{(i_{1,\frac{nd}{2}},\gamma_{i_{1,\frac{nd}{2}}}(j_{1,\frac{nd}{2}})),(i_{2,\frac{nd}{2}},\gamma_{i_{2,\frac{nd}{2}}}(j_{2,\frac{nd}{2}}))\}\Big\} \quad . \end{aligned} \tag{11}$$

Note that in (17) the first component of each pair determines which of the n permutations $\gamma_1,\dots,\gamma_n$ to use for the second pair. Let us now take the time to prove the correctness of this definition: since by definition the multiplication in $\times^n\mathfrak{S}_{[d]}$ is componentwise, we have $(\gamma'\gamma'')_i = \gamma'_i\gamma''_i$ and hence for arbitrary $\gamma',\gamma'' \in \Gamma$ we have $\alpha(\gamma'\gamma'') = \alpha(\gamma')\alpha(\gamma'')$, i.e. α is indeed a group-homomorphism. What is not that obvious is that the $\alpha(\gamma)(C)$ defined in (17) actually is a configuration.

Lemma 10 (correctness of codomain of α)**.** *For every even* $nd \ge 2$*, every* $\gamma = (\gamma_1,\dots,\gamma_n) \in \Gamma$*, and*

(1) $|\{(i_{1,\kappa},\gamma_{i_{1,\kappa}}(j_{1,\kappa})),(i_{2,\kappa},\gamma_{i_{2,\kappa}}(j_{2,\kappa}))\}| = 2$ *for every* $1 \leq \kappa \leq \frac{nd}{2}$,

(2) $\{(i_{1,\kappa_1},\gamma_{i_{1,\kappa_1}}(j_{1,\kappa_1})),(i_{2,\kappa_1},\gamma_{i_{2,\kappa_1}}(j_{2,\kappa_1}))\} \cap \{(i_{1,\kappa_2},\gamma_{i_{1,\kappa_2}}(j_{1,\kappa_2})),(i_{2,\kappa_2},\gamma_{i_{2,\kappa_2}}(j_{2,\kappa_2}))\} = \emptyset$ *for every* $1 \leq \kappa_1 < \kappa_2 \leq \frac{nd}{2}$,

(3) *The set* $\alpha(\gamma)(C)$ *in Definition 9.*(10) *is a partition of* $[n]\times[d]$ *into blocks of size* 2*, i.e. an element of* $\mathrm{Conf}_{[n]\times[d]}$.

Proof of Lemma 10. As to (1), we argue as follows: if $i_{1,\kappa} \neq i_{2,\kappa}$, then this non-equality by itself implies $(i_{1,\kappa},\gamma_{i_{1,\kappa}}(j_{1,\kappa})) \neq (i_{2,\kappa},\gamma_{i_{2,\kappa}}(j_{2,\kappa}))$, i.e. (1). If on the contrary $i_{1,\kappa} = i_{2,\kappa} =: i_\kappa$, then since we defined the set in (10) to be a configuration in the definition of α, we know that $j_{1,\kappa} \neq j_{2,\kappa}$. This implies that $\gamma_{i_{1,\kappa}}(j_{1,\kappa}) = \gamma_{i_\kappa}(j_{1,\kappa}) \neq$ (since $j_{1,\kappa} \neq j_{2,\kappa}$ and γ_{i_κ} is a permutation) $\neq \gamma_{i_\kappa}(j_{2,\kappa}) = \gamma_{i_{2,\kappa}}(j_{2,\kappa})$, hence indeed $(i_{1,\kappa},\gamma_{i_{1,\kappa}}(j_{1,\kappa})) \neq (i_{2,\kappa},\gamma_{i_{2,\kappa}}(j_{2,\kappa}))$, i.e. (1).

As to (2), suppose there were $1 \leq \kappa_1 < \kappa_2 \leq \frac{nd}{2}$ for which this is false. Not having made any demands on the letters involved, we may assume without loss of generality that disjointness fails because of $(i_{1,\kappa_1},\gamma_{i_{1,\kappa_1}}(j_{1,\kappa_1})) = (i_{1,\kappa_2},\gamma_{i_{1,\kappa_2}}(j_{1,\kappa_2}))$. Then $i_{1,\kappa_1} = i_{1,\kappa_2} =: i_1$ and $\gamma_{i_{1,\kappa_1}}(j_{1,\kappa_1}) = \gamma_{i_{1,\kappa_2}}(j_{1,\kappa_2})$, hence $\gamma_{i_1}(j_{1,\kappa_1}) = \gamma_{i_{1,\kappa_1}}(j_{1,\kappa_1}) = \gamma_{i_{1,\kappa_2}}(j_{1,\kappa_2}) = \gamma_{i_1}(j_{1,\kappa_2})$, hence $j_{1,\kappa_1} = j_{1,\kappa_2}$ by applying $\gamma_{i_1}^{-1}$, and hence $\{(i_{1,\kappa_1},j_{1,\kappa_1}),(i_{2,\kappa_1},j_{2,\kappa_1})\} \cap \{(i_{1,\kappa_2},j_{1,\kappa_2}),(i_{2,\kappa_2},j_{2,\kappa_2})\} \supseteq \{(i_{1,\kappa_1},j_{1,\kappa_2})\} \neq \emptyset$, contradicting the hypothesis that C is a configuration. This proves (2).

As to (3), this follows immediately from (1) and (2): the two statements say that $\alpha(\gamma)(C)$ is a set of $\frac{nd}{2}$ mutually disjoint 2-element subsets of $[n]\times[d]$. □

We recall that if a group Γ acts via a homomorphism $\alpha\colon \Gamma \to \mathfrak{S}_\Omega$ on a set Ω, then for every $\omega \in \Omega$ the *stabilizer of* ω *w.r.t. to* α is defined as $\mathrm{Stab}_\alpha(\omega) := \{\gamma \in \Gamma\colon \alpha(\gamma)(\omega) = \omega\}$ and for every $\omega \in \Omega$ the *orbit of* ω *under* α is defined as $\Gamma \cdot_\alpha \omega := \{\alpha(\gamma)(\omega)\colon \gamma \in \Gamma\}$. The *orbit-stabilizer-formula*[2] says that

$$|\Gamma \cdot_\alpha \omega| = \frac{|\Gamma|}{|\mathrm{Stab}_\alpha(\omega)|} \quad . \tag{12}$$

It is very common in the literature not to reflect the group action α in the notation.

Our goal is to determine, for a given multigraph G, the cardinality of the finite sets of the form $\pi^{-1}(G)$. To this end, we will employ the orbit stabilizer formula and our group-action α from Definition 9.

Let us first speak in general terms. Suppose you would like to find the cardinality of a finite set Ω with the help of the orbit-stabilizer-formula. What you can of course always do is to take Γ to be the symmetric group on Ω; but this is a 'degenerate' case of the orbit-stabilizer-formula which merely tells us that for an arbitrary $\omega \in \Omega$,

$$|\Omega| = \mathfrak{S}_\Omega \cdot \omega = \frac{|\mathfrak{S}_\Omega|}{|\mathrm{Stab}_\alpha(\omega)|} = \frac{|\Omega|!}{(|\Omega|-1)!} \quad , \tag{13}$$

and because of the explicit dependence of the right-hand-side on $|\Omega|$, which we do not know, we have gained nothing. This is nothing more than exhaustive enumeration: we shuffled the elements of ω around as opaque points and did not make use of the eveloping structure.

The art in using the orbit-stabilizer lies in making a choice of a group Γ which is much smaller than $\mathfrak{S}_\Omega$ and which must reconcile two conflicting demands:

(D.1) the group Γ via α acts transitively on Ω ,

[2]This is nothing deep: by elementary group theory, right-hand side is cardinality of the factor group $\Gamma/\mathrm{Stab}_\alpha(\omega)$; moreover $\alpha(\gamma)(\omega) \mapsto \gamma \cdot \mathrm{Stab}_\alpha(\omega)$ is easily shown to unambiguously define a bijection $\Gamma \cdot_\alpha \omega \longleftrightarrow \Gamma/\mathrm{Stab}_\alpha(\omega)$.

(D.2) the number $\frac{|\Gamma|}{|\mathrm{Stab}_\alpha(\omega)|} = |G/\mathrm{Stab}_\alpha(\omega)|$ can be determined without knowing $|\Omega|$.

Demand (D.1) is a condition of 'clinging to the objects we want to count' and the group $\mathfrak{S}_\Omega$ trivially meets this criterion. Demand (D.2) is a condition of 'letting go of the objects and manipulating the surrounding structure' and the group $\mathfrak{S}_\Omega$ trivially fails this criterion.

We will prove that our α from Definition 9 satisfies both (D.1) and (D.2). We start with (D.2).

Lemma 11 *For every even $nd \geq 2$ and every $G \in \mathrm{Mult}_{n,d}$, the action of Γ via α on $\pi^{-1}(G) \subset \mathrm{Conf}_{[n]\times[d]}$ is transitive.*

Proof of Lemma 11. We denote by $C' = \{ \{ (\imath'_{1,1}, \jmath'_{1,1}), (\imath'_{2,1}, \jmath'_{2,1}) \}, \{ (\imath'_{1,2}, \jmath'_{1,2}), (\imath'_{2,2}, \jmath'_{2,2}) \}, \{ (\imath'_{1,3}, \jmath'_{1,3}), (\imath'_{2,3}, \jmath'_{2,3}) \}, \dots, \{ (\imath'_{1,\frac{nd}{2}}, \jmath'_{1,\frac{nd}{2}}), (\imath'_{2,\frac{nd}{2}}, \jmath'_{2,\frac{nd}{2}}) \} \}$ and $C'' = \{ \{ (\imath''_{1,1}, \jmath''_{1,1}), (\imath''_{2,1}, \jmath''_{2,1}) \}, \{ (\imath''_{1,2}, \jmath''_{1,2}), (\imath''_{2,2}, \jmath''_{2,2}) \}, \{ (\imath''_{1,3}, \jmath''_{1,3}), (\imath''_{2,3}, \jmath''_{2,3}) \}, \dots, \{ (\imath''_{1,\frac{nd}{2}}, \jmath''_{1,\frac{nd}{2}}), (\imath''_{2,\frac{nd}{2}}, \jmath''_{2,\frac{nd}{2}}) \} \}$ arbitrary elements of $\pi^{-1}(G) \subset \mathrm{Conf}_{[n]\times[d]}$, i.e. $\pi(C') = \pi(C'') = G$, written in such a way that the labellings of C' and C'' satisfy (Ord.1), (Ord.2) and (Ord.3) in Definition 6. We have to prove the existence of a $\gamma \in \times^n \mathfrak{S}_{[d]}$ with $\alpha(\gamma)(C') = C''$. We claim that the assignments

(A.1) $\gamma_{\imath'_{1,\kappa}}(\jmath'_{1,\kappa}) := \jmath''_{1,\kappa}$ for every $1 \leq \kappa \leq \frac{nd}{2}$,

(A.2) $\gamma_{\imath'_{2,\kappa}}(\jmath'_{2,\kappa}) := \jmath''_{2,\kappa}$ for every $1 \leq \kappa \leq \frac{nd}{2}$,

define such a γ. To justify this, we have to say why

(1) this does indeed define an element of $\times^n \mathfrak{S}_{[d]}$,

(2) $\alpha(\gamma)(C') = C''$.

As to (1) since $C' \in \mathrm{Conf}_{[n]\times[d]}$, we know that $j'_{1,\kappa} \neq j'_{2,\kappa}$ for every $1 \leq \kappa \leq \frac{nd}{2}$, hence the assignments are at least unambiguous. If for every $i \in [n]$ we define

$$K_i := \{\kappa \in \{1, \dots, \tfrac{nd}{2}\} \colon \ \imath'_{1,\kappa} = i\} \,\dot{\cup}\, \{\kappa \in \{1, \dots, \tfrac{nd}{2}\} \colon \ \imath'_{2,\kappa} = i\} \quad , \tag{14}$$

then the assumption that $C' \in \mathrm{Conf}_{[n]\times[d]}$ implies $K_i \neq \emptyset$ for every $i \in [n]$, and also

$$\{\jmath'_{1,\kappa} \colon \kappa \in K_i\} \,\dot{\cup}\, \{\jmath'_{2,\kappa} \colon \kappa \in K_i\} \subset [d] \quad , \tag{15}$$

hence (A.1) and (A.2) define an element of $\times^n [d]^{[d]}$. Since C'' is assumed to be an element of $\mathrm{Conf}_{[n]\times[d]}$ we moreover know that for every $i \in [n]$ and arbitrary $\kappa_1 \neq \kappa_2 \in K_i$,

(1) $j''_{1,\kappa_1} \neq j''_{1,\kappa_2}$,

(2) $j''_{2,\kappa_1} \neq j''_{2,\kappa_2}$,

(3) $j''_{1,\kappa_1} \neq j''_{2,\kappa_2}$.

Therefore (A.1) and (A.2) do in fact define an element of $\times^n \mathfrak{S}_{[d]}$ (in particular, (15) holds with $=$ instead of $\subset$).

As to (2), because of $\pi(C') = \pi(C'')$ we know $\mathrm{g}^\pi_{C'} = \mathrm{g}^\pi_{C''}$, hence

$$\{\imath'_{1,\kappa}, \imath'_{2,\kappa}\} = \mathrm{g}^\pi_{C'}(e^{n,d}_\kappa) = \mathrm{g}^\pi_{C''}(e^{n,d}_\kappa) = \{\imath''_{1,\kappa}, \imath''_{2,\kappa}\} \quad , \tag{16}$$

and therefore

$$\begin{aligned}
\alpha(\gamma)(C') :=& \Big\{\{(\imath'_{1,1}, \gamma_{\imath'_{1,1}}(\jmath'_{1,1})), (\imath'_{2,1}, \gamma_{\imath'_{2,1}}(\jmath'_{2,1}))\}, \\
& \{(\imath'_{1,2}, \gamma_{\imath'_{1,2}}(\jmath'_{1,2})), (\imath'_{2,2}, \gamma_{\imath'_{2,2}}(\jmath'_{2,2}))\}, \\
& \{(\imath'_{1,3}, \gamma_{\imath'_{1,3}}(\jmath'_{1,3})), (\imath'_{2,3}, \gamma_{\imath'_{2,3}}(\jmath'_{2,3}))\}, \\
& \ldots, \\
& \{(\imath'_{1,\frac{nd}{2}}, \gamma_{\imath'_{1,\frac{nd}{2}}}(\jmath'_{1,\frac{nd}{2}})), (\imath'_{2,\frac{nd}{2}}, \gamma_{\imath'_{2,\frac{nd}{2}}}(\jmath'_{2,\frac{nd}{2}}))\}\Big\} \\
\text{(by (A.1) and (A.2))} \quad =& \Big\{\{(\imath'_{1,1}, \jmath''_{1,1}), (\imath'_{2,1}, \jmath''_{2,1})\}, \\
& \{(\imath'_{1,2}, \jmath''_{1,2}), (\imath'_{2,2}, \jmath''_{2,2})\}, \\
& \{(\imath'_{1,3}, \jmath''_{1,3}), (\imath'_{2,3}, \jmath''_{2,3})\}, \\
& \ldots, \\
& \{(\imath'_{1,\frac{nd}{2}}, \jmath''_{1,\frac{nd}{2}}), (\imath'_{2,\frac{nd}{2}}, \jmath''_{2,\frac{nd}{2}})\}\Big\} \\
\text{(by (16))} \quad =& \Big\{\{(\imath''_{1,1}, \jmath''_{1,1}), (\imath''_{2,1}, \jmath''_{2,1})\}, \\
& \{(\imath''_{1,2}, \jmath''_{1,2}), (\imath''_{2,2}, \jmath''_{2,2})\}, \\
& \{(\imath''_{1,3}, \jmath''_{1,3}), (\imath''_{2,3}, \jmath''_{2,3})\}, \\
& \ldots, \\
& \{(\imath''_{1,\frac{nd}{2}}, \jmath''_{1,\frac{nd}{2}}), (\imath''_{2,\frac{nd}{2}}, \jmath''_{2,\frac{nd}{2}})\}\Big\} \\
\text{(by definition of } C'') \quad =& \quad C'' \quad , \qquad (17)
\end{aligned}$$

proving (2) and completing the proof of Lemma 11.

Definition 12 (loop-degree of an $i \in [n]$ within a configuration, multi-edge-degree of an $\{i_1, i_2\} \in \binom{[n]}{2}$ within a configuration)*. Let* $nd \geq 2$ *be even. Let* $C = \{\ \{(\imath_{1,1}, \jmath_{1,1}),\ (\imath_{2,1}, \jmath_{2,1})\},\ \{(\imath_{1,2}, \jmath_{1,2}),\ (\imath_{2,2}, \jmath_{2,2})\},\ \{(\imath_{1,3}, \jmath_{1,3}),\ (\imath_{2,3}, \jmath_{2,3})\},\ \ldots,\ \{(\imath_{1,\frac{nd}{2}}, \jmath_{1,\frac{nd}{2}}),\ (\imath_{2,\frac{nd}{2}}, \jmath_{2,\frac{nd}{2}})\}\ \}$ *denote an arbitrary element of* $\mathrm{Conf}_{[n]\times[d]}$*, written in such a way that* (Ord.1)*,* (Ord.2) *and* (Ord.3) *in Definition 6 are satisfied. Then we define:*

(0) *the* loop-degree of i in C *as* $\mathrm{ld}_C(i) := |\{\kappa \in [\frac{nd}{2}] \colon \{\imath_{1,\kappa}, \imath_{2,\kappa}\} = \{i\}\}| \in \mathbb{Z}$ *for every* $i \in [n]$,

(1) *the* multi-edge-degree of $\{i_1, i_2\} \in \binom{[n]}{2}$ in C *as* $\mathrm{md}_C(i_1, i_2) := |\{\kappa \in [\frac{nd}{2}] \colon \{\imath_{1,\kappa}, \imath_{2,\kappa}\} = \{i_1, i_2\}\}|$ *for every* $\{i_1, i_2\} \in \binom{[n]}{2}$.

In the terminology of Definition 4 a set $\{i_1, i_2\} \in \binom{[n]}{2}$ is the ends of a multi-edge if and only if $\mathrm{md}_C(i_1, i_2) > 1$.

Lemma 13 (size of stabilizer subgroup of a configuration w.r.t. the action α)*. Let* $nd \geq 2$ *be even. Let* $C = \{\ \{(\imath_{1,1}, \jmath_{1,1}),\ (\imath_{2,1}, \jmath_{2,1})\},\ \{(\imath_{1,2}, \jmath_{1,2}),(\imath_{2,2}, \jmath_{2,2})\},\ \{(\imath_{1,3}, \jmath_{1,3}),(\imath_{2,3}, \jmath_{2,3})\},\ \ldots,\ \{(\imath_{1,\frac{nd}{2}}, \jmath_{1,\frac{nd}{2}}),$

$(\imath_{2,\frac{nd}{2}}, \jmath_{2,\frac{nd}{2}})\}$ } *denote an arbitrary element of* $\mathrm{Conf}_{[n]\times[d]}$*, written in such a way that* (Ord.1)*,* (Ord.2) *and* (Ord.3) *in Definition 6 are satisfied. Then*

$$|\mathrm{Stab}_\alpha(C)| = \left(\prod_{i\in[n]} (2^{\mathrm{ld}_C(i)} \cdot \mathrm{ld}_C(i)!)\right) \cdot \left(\prod_{\{i_1,i_2\}\in\binom{[n]}{2}} \mathrm{md}_C(i_1,i_2)!\right) \quad . \tag{18}$$

Proof of Lemma 13. We leave the formal work as an exercise for you. We intentionally chose this proof to leave to you: the claim is a very constrained one, which should make this task rather easy (remember that the elements of $\mathrm{Stab}_\alpha(C)$ are those $\gamma \in \times^n \mathfrak{S}_{[d]}$ which via α leave C *exactly equal*, as a set of 2-sets of pairs). This strong requirement should enable you to prove that there are *not more* than $(\prod_{i\in[n]}(2^{\mathrm{ld}_C(i)} \cdot \mathrm{ld}_C(i)!)) \cdot (\prod_{\{i_1,i_2\}\in\binom{[n]}{2}} \mathrm{md}_C(i_1,i_2)!)$, elements in $\mathrm{Stab}_\alpha(C)$, which is a crucial thing to prove when one would like to solve this problem rigorously. To see that there are *at least* the claimed number of elements in $\mathrm{Stab}_\alpha(C)$, note the following:

(1) For every $\{i_1,i_2\} \in \binom{[n]}{2}$ each of the $\mathrm{md}_C(i_1,i_2)!$ permutations of the set of second components of second pairs in the 2-set of pairs comprising the multi-edge can be extended to an element of $\mathrm{Stab}_\alpha(C)$ by choosing the corresponding permutation of the set of second components of first pairs.

(2) For every $i \in [n]$ the (possibly empty) set of loop having i as their end-vertex defines set of mutually disjoint 2-element subsets of $[d]$ and every permutation of $[d]$ which in any way permutes these 2-sets as a whole and at the same time does or does not switch the two elements the 2-set results in exactly the same configuration; hence the factor $(\prod_{i\in[n]}(2^{\mathrm{ld}_C(i)} \cdot \mathrm{ld}_C(i)!))$. The 'permuting the 2-sets as a whole' part could also be phrased as '*permuting multiple loops at a vertex among each other*'. This possibility seems to have been overlooked in certain places in the literature.

□

Lemma 14 (configurations mapping to a given multigraph have the loop- and multi-edge-degrees of this multigraph)**.** *For every even* $nd \geq 2$*, every* $G \in \mathrm{Mult}_{n,d}$*, every* $i \in [n]$*, every* $\{i_1,i_2\} \in \binom{[n]}{2}$ *and every* $C \in \pi^{-1}(G)$ *we have* $\mathrm{ld}_C(i) = \mathrm{ld}_G(i)$ *and* $\mathrm{md}_C(i_1,i_2) = \mathrm{md}_G(i_1,i_2)$*.*

Proof of Lemma 14. Left to you as an exercise of writing a proof within the formalism we set up so far. □

We can now give a solution for Problem 5.1:

Proposition 15 (number of configurations giving rise to a given multigraph)**.** *For every even* $nd \geq 2$ *and every* $G \in \mathrm{Mult}_{n,d}$*,*

$$|\pi^{-1}(G)| = \frac{(d!)^n}{\left(\prod_{i\in[n]}(2^{\mathrm{ld}_G(i)} \cdot \mathrm{ld}_G(i)!)\right) \cdot \left(\prod_{\{i_1,i_2\}\in\binom{[n]}{2}} \mathrm{md}_G(i_1,i_2)!\right)} \quad . \tag{19}$$

Proof of Proposition 15. Let $C \in \pi^{-1}(G)$ be an arbitrary element. Then

$$
\begin{aligned}
|\pi^{-1}(G)| &\overset{\text{(by Lemma 11)}}{=} |(\times^n \mathfrak{S}_{[d]}) \cdot_\alpha C| \\
\text{(by the orbit-stabilizer-formula (12))} &= \frac{|\times^n \mathfrak{S}_{[d]}|}{|\mathrm{Stab}_\alpha(C)|} = \frac{(d!)^n}{|\mathrm{Stab}_\alpha(C)|} \\
\text{(by Lemma 13)} &= \frac{(d!)^n}{\left(\prod_{i\in[n]} (2^{\mathrm{ld}_C(i)} \cdot \mathrm{ld}_C(i)!)\right) \cdot \left(\prod_{\{i_1,i_2\}\in\binom{[n]}{2}} \mathrm{md}_C(i_1,i_2)!\right)} \\
\text{(by Lemma 14)} &= \frac{(d!)^n}{\left(\prod_{i\in[n]} (2^{\mathrm{ld}_G(i)} \cdot \mathrm{ld}_G(i)!)\right) \cdot \left(\prod_{\{i_1,i_2\}\in\binom{[n]}{2}} \mathrm{md}_G(i_1,i_2)!\right)} \quad , \qquad (20)
\end{aligned}
$$

as claimed. □

An example.

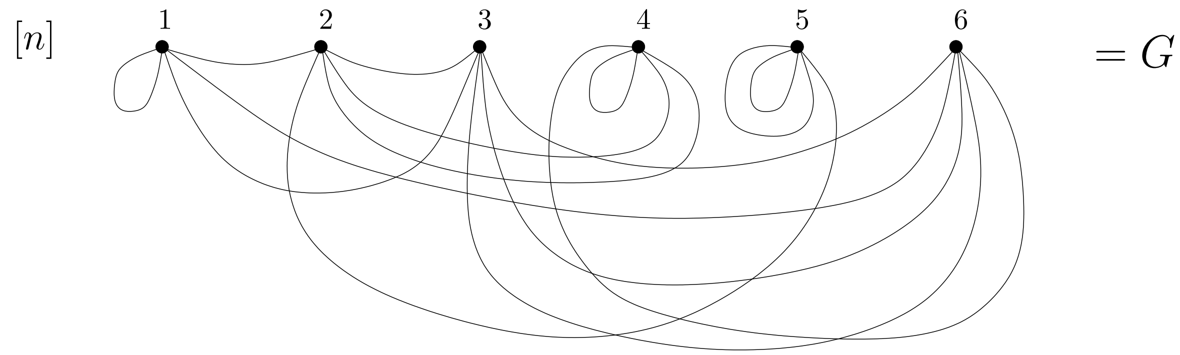

Figure 1: An example with $n = 6$ and $d = 5$: this visualizes a 5-regular multigraph on 6 vertices.

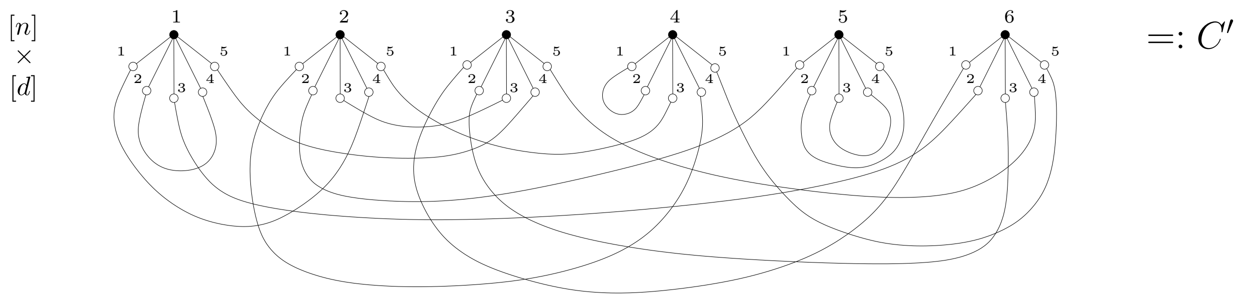

Figure 2: This visualizes a configuration C' in the preimage of the G from Figure 1 w.r.t. π. The stabilizer subgroup of C' in $\times^6 \mathfrak{S}_{[5]}$ w.r.t. the action α from Definition 9 has exactly $(1! \cdot 2^1 \cdot 1! \cdot 2^1 \cdot 2! \cdot 2^2) \cdot (2! \cdot 3!) = 384$ elements. (First bracketed factor due to loops, second bracketed factor due to multiple edges.)

The configuration C' from Figure 2, written so as to make (Ord.1), (Ord.2) and (Ord.3) in Definition 6 hold true, is $\{\ \{(1,2),(1,4)\},\ \{(1,1),(2,4)\},\ \{(1,5),(3,4)\},\ \{(1,3),(6,2)\},\ \{(2,3),(3,3)\},$ $\{(2,1),(4,4)\},\ \{(2,5),(4,3)\},\ \{(2,2),(5,1)\},\ \{(3,1),(6,1)\},\ \{(3,2),(6,3)\},\ \{(3,5),(6,4)\},\ \{(4,1),(4,2)\},$ $\{(4,5),(6,5)\},\ \{(5,2),(5,5)\},\ \{(5,3),(5,4)\}\ \}$.

The configuration C'' from Figure 3, written so as to make (Ord.1), (Ord.2) and (Ord.3) in Def-

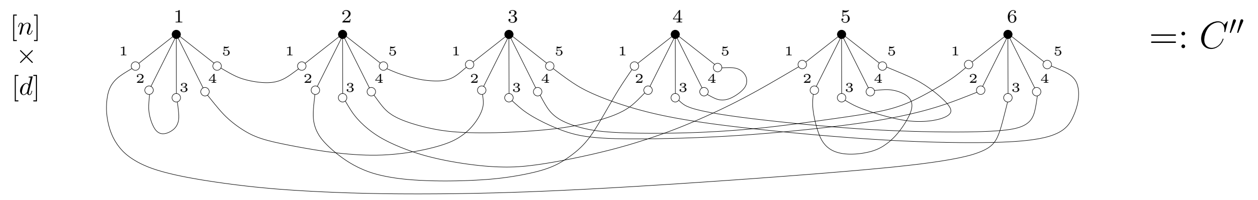

Figure 3: This visualizes another configuration C'' in the preimage of the G from Figure 1 w.r.t. π. The stabilizer subgroup of C'' in $\times^6\mathfrak{S}_{[5]}$ w.r.t. the action α from Definition 9 has, just as for C', exactly $(1! \cdot 2^1 \ \cdot\ 1! \cdot 2^1 \ \cdot\ 2! \cdot 2^2) \cdot (2! \cdot 3!) = 384$ elements.

inition 6 hold true, is $\{\ \{(1,2),(1,3)\},\ \{(1,5),(2,1)\},\ \{(1,4),(3,2)\},\ \{(1,1),(6,3)\},\ \{(2,5),(3,1)\},$ $\{(2,2),(4,1)\}, \{(2,4),(4,2)\}, \{(2,3),(5,1)\}, \{(3,3),(6,2)\}, \{(3,4),(6,1)\}, \{(3,5),(6,5)\}, \{(4,4),(4,5)\},$ $\{(4,3),(6,4)\},\ \{(5,2),(5,4)\},\ \{(5,3),(5,5)\}\ \}$.

For these C' and C'', the element $\gamma \in \times^6\mathfrak{S}_{[5]}$ with $\alpha(\gamma)(C') = C''$ defined by (A.1) and (A.2) is

$$\gamma := \Big(\left(\begin{smallmatrix}1&2&3&4&5\\5&2&1&3&4\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3&4&5\\2&3&5&1&4\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3&4&5\\3&4&1&2&5\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3&4&5\\4&5&2&1&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3&4&5\\1&2&3&5&4\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3&4&5\\2&3&1&5&4\end{smallmatrix}\right) \Big) \quad \in \quad \times^6\mathfrak{S}_{[5]} \quad . \tag{21}$$