Random Graphs (MA 5208) Summer 2012
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Sheet 6

Problem 6.1 *On chromatic number when the expected degree is larger than one.*
Prove that for every $\ell \in \mathbb{Z}_{\geq 3}$ there is $c_\ell > 1$ so that $p_n \geq \frac{c_\ell}{n}$ implies $\mathsf{Pr}_{G(n,p_n)}\Big[\chi(G) \leq \ell\Big] \xrightarrow{n\to\infty} 0$.

A solution. We will use the following inequality:

Lemma 1 *For every $(a_1,\dots,a_\ell) \in \mathbb{R}^\ell_{\geq 0}$ with $\sum_{1\leq i\leq \ell} a_i = n$ we have $\sum_{1\leq i\leq \ell}\binom{a_i}{2} \geq \ell\binom{\frac{n}{\ell}}{2}$.*

Proof of Lemma 1. Because of $\sum_{1\leq i\leq\ell} a_i = n$ the statement $\sum_{1\leq i\leq\ell}\binom{a_i}{2} \geq \ell\binom{\frac{n}{\ell}}{2}$ is equivalent to $\sum_{1\leq i\leq\ell} a_i^2 \geq \frac{n^2}{\ell}$ which is true because $n^2 = (\sum_{1\leq i\leq\ell} a_i)^2 = \langle(a_1,\dots,a_\ell),(1,\dots,1)\rangle^2 \leq$ (Bunyakovsky-Cauchy-Schwarz) $\leq \|(1,\dots,1)\|_2^2 \cdot \|(a_1,\dots,a_\ell)\|_2^2 = \ell \cdot \sum_{1\leq i\leq\ell} a_i^2$. □

Now Problem 6.1 can be solved with the first-moment-method, as follows: let $\ell \in \mathbb{Z}_{\geq 3}$ be given. Choose any $c_\ell > 1$ with the property that $\mathrm{b}_\ell := \ell \cdot \exp(-\frac{1}{2}c_\ell/\ell) < 1$. Define $\mathrm{X}\colon \mathcal{G}_n \longrightarrow \mathbb{Z}_{\geq 0}$ by $\mathrm{X}(G) := |\{\, \mathrm{c} \in [\ell]^{[n]}\colon \binom{\mathrm{c}^{-1}(a)}{2} \cap \mathrm{E}(G) = \emptyset \text{ for every } a \in [\ell]\,\}|$. Abbreviate $\mathsf{Pr} := \mathsf{Pr}_{G(n,p_n)}$ and $\mathsf{Ex} := \mathsf{Ex}_{G(n,p_n)}$ Then for every n large enough to make both $1 - \frac{c_\ell}{n} \leq \exp(-\frac{c_\ell}{n})$ and $\exp(-c_\ell(\frac{n}{\ell}-1)) \leq \exp(-\frac{1}{2}c_\ell\frac{n}{\ell})$ simultaneously true we have $\mathsf{Pr}\,[\,\{\, G \in \mathcal{G}_n\colon \chi(G) \leq \ell \,\}\,] = \mathsf{Pr}\,[\,\{\, G \in \mathcal{G}_n\colon \mathrm{X}(G) \geq 1 \,\}\,] \leq$ (Markov) $\leq \mathsf{Ex}\,[\,\mathrm{X}\,] = \sum_{\mathrm{c}\in[\ell]^{[n]}} \mathsf{Pr}\,[\,\binom{\mathrm{c}^{-1}(a)}{2} \cap \mathrm{E}(G) = \emptyset \text{ for every } a \in [\ell]\,] = \sum_{\mathrm{c}\in[\ell]^{[n]}} (1-p_n)^{\sum_{a\in[\ell]}\binom{\mathrm{c}^{-1}(a)}{2}}$ $\leq$ (because of $(1-p_n) < 1$ and $\sum_{a\in[\ell]}|\mathrm{c}^{-1}(a)| = n$ and Lemma 1) $\leq \ell^n\,(1-p_n)^{\ell\cdot\binom{\frac{n}{\ell}}{2}} \leq$ (since $1-\varepsilon \leq \exp(-\varepsilon)$ for sufficiently small $\varepsilon > 0$) $\leq \ell^n \cdot \exp(-\frac{c_\ell}{n})^{\ell\cdot\binom{\frac{n}{\ell}}{2}} \leq \ell^n \cdot \exp(-c_\ell(\frac{n}{\ell}-1)) \leq$ (by choice of n) $\leq \ell^n \cdot \exp(-\frac{1}{2}c_\ell\frac{n}{\ell}) = (\ell\cdot\exp(-\frac{1}{2}c_\ell/\ell))^n <$ (by choice of c_ℓ) $< (\mathrm{b}_\ell)^n \xrightarrow{n\to\infty} 0$, completing the solution of Problem 6.1.

Problem 6.2 *Chernoff-bounds, and concentration of the number of edges.*
Prove that if X is binom(n,p)-distributed, then for every $\delta \geq 0$,

$$\mathsf{Pr}[|X - \mathsf{Ex}[X]| \geq \delta] \leq \exp(-\tfrac{\delta^2}{2\cdot\mathsf{Ex}[X]}) \quad . \tag{1}$$

Now show that Chebyshev's inequality gives the estimate that for every *constant* $0 < p < 1$ there exists a constant $C_p > 0$ such that

$$\mathsf{Pr}_{G(n,p)}\Big[\Big|\|\cdot\| - \mathsf{Ex}[\|\cdot\|]\Big| > n^{3/2}\Big] \leq \frac{C_p}{n} \quad . \tag{2}$$

whereas (1) tells us that for every constant $0 < p < 1$ there exists a constant $C_p > 0$ with

$$\mathsf{Pr}_{G(n,p)}\Big[\Big|\|\cdot\| - \mathsf{Ex}[\|\cdot\|]\Big| > n^{3/2}\Big] \leq \exp(-\tfrac{n}{C_p}) \quad . \tag{3}$$

A solution. A general method for deriving concentration bounds works by introducing an auxiliary indeterminate, as follows: for every $u > 0$ we have $\mathsf{Pr}[X \geq \mathsf{Ex}[X] + t] = \mathsf{Pr}[\exp(uX) \geq \exp(u(\mathsf{Ex}[X] +$

$t))] \leq$ (Markov) $\leq \exp(-u(\mathsf{Ex}[X]+t)) \cdot \mathsf{Ex}[\exp(uX)]$. For further reference:

$$\mathsf{Pr}[X \geq \mathsf{Ex}[X]+t] \leq \exp(-u(\mathsf{Ex}[X]+t)) \cdot \mathsf{Ex}[\exp(uX)] \quad . \tag{4}$$

It may happen that, when a specific distribution of X is given, the right-hand side of (4) can be optimized over $u \in \mathbb{R}_{>0}$. Then substituting a minimizing $u \in \mathbb{R}_{>0}$, so as to eke out the most out of (4), may lead to useful bounds.

It is by this method that one can derive (1).

Since in Problem 6.2 we know that $\mathsf{Ex}[X] = np$ we can calculate $\mathsf{Pr}[X \geq \mathsf{Ex}[X]+\delta] \leq$ (by (4)) $\leq \exp(-u(np+\delta)) \cdot \mathsf{Ex}[\exp(uX)] =$ (because of $X = \sum_{1\leq i\leq n} X_i$ with $X_i \sim \text{Bernoulli}(p)$) $= \exp(-u(np+\delta)) \cdot \mathsf{Ex}[\prod_{1\leq i\leq n} \exp(uX_i)] =$ (since $\exp(uX_1), \ldots, \exp(uX_n)$ are independent) $= \exp(-u(np+\delta)) \cdot \prod_{1\leq i\leq n} \mathsf{Ex}[\exp(uX_i)] = \exp(-u(np+\delta)) \cdot ((1-p)+p\exp(u))^n =: g(u)$ and now we are interested in finding the minimum of $g(u)$. A calculation shows that

$$\mathrm{D}_u g(u) = 0 \qquad \Leftrightarrow \qquad \exp(u) = 1 + \frac{\delta}{(n-\delta-np)p} \quad . \tag{5}$$

We may assume that $n - np - \delta > 0$, for $X \sim \text{binom}(n,p)$ implies that the negation of that consists of the trivial situations $\delta = n - np$, hence $\mathsf{Pr}[X \geq \mathsf{Ex}[X]+\delta] = \mathsf{Pr}[X = n] = p^n$, or $\delta > n - np$, hence $\mathsf{Pr}[X \geq \mathsf{Ex}[X]+\delta] = 0$. Because of $\delta < n - np$ we know that $1 + \frac{t}{(n-\delta-np)p} > 0$, hence equation (5) has exactly one solution:

$$u_{\mathrm{m}} = \log\left(1 + \frac{\delta}{(n-\delta-np)p}\right) \quad . \tag{6}$$

Calculus shows that u_{m} is a minimum of $g(u)$, and now after substituting u with u_{m} leads to (1).

As to (2), note that $\|\cdot\|$ is the sum of $\binom{n}{2}$ independent bernoulli(p)-distributed random variables with image $\{0,1\}$. Since variance is linear for independent variables we therefore immediately get $\mathsf{Var}[\|\cdot\|] = p(1-p)\binom{n}{2}$ and can use Chebychev's inequality as $\mathsf{Pr}_{\mathrm{G}(n,p)}\left[\,\left|\,\|\cdot\| - \mathsf{Ex}[\|\cdot\|]\,\right| > n^{3/2}\,\right] \leq \frac{\mathsf{Var}[\|\cdot\|]}{(n^{3/2})^2} = \frac{\binom{n}{2}p(1-p)}{n^3} \leq \frac{n^2p(1-p)}{n^3} = \frac{p(1-p)}{n}$, which proves (2) with $C_p := (1-p)p$.

As to (1), an equivalent way of phrasing the comments at the beginning of the preceding paragraph is to say that $\|\cdot\|$ is binomial$(\binom{n}{2}, p)$-distributed. Therefore we may use (1) with $X := \|\cdot\|$, $\mathsf{Ex}[X] = \binom{n}{2}p$ and $\delta := n^{3/2}$ to get $\mathsf{Pr}_{\mathrm{G}(n,p)}\left[\,\left|\,\|\cdot\| - \mathsf{Ex}[\|\cdot\|]\,\right| > n^{3/2}\,\right] \leq \exp(-\frac{n^3}{2\cdot\binom{n}{2}p}) <$ (for $n > 2$) $< \exp(-\frac{n^3}{\frac{1}{2}n^2p})$ $= \exp(-\frac{n}{\frac{1}{2}p})$, which proves (1) with $C_p := \frac{1}{2}p$.

Problem 6.3 *An aspect of the phase transition for connectedness: isolated vertices.*
Prove that for every $c \in \mathbb{R}$,

$$\mathsf{Pr}_{\mathrm{G}(n,\frac{c+\log n}{n})}[\{G \in \mathcal{G}_n : \text{in } G \text{ there does not exist an isolated vertex}\}] \xrightarrow{n\to\infty} \exp(-\exp(-c)) \quad . \tag{7}$$

A solution.

Lemma 2 *If $(p_n) \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ with $n \cdot p_n^2 \xrightarrow{n\to\infty} 0$, then $(1-p_n)^n \sim \exp(-p_n \cdot n)$.*

Proof of Lemma 2. The continuity of log implies that for sequences $(f_n), (g_n) \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ the property $\lim \frac{f_n}{g_n} = 1$ is equivalent to $\lim \log \frac{f_n}{g_n} = 0$. To prove the latter, we consider $\log \frac{(1-p_n)^n}{\exp(-p_n n)} = n\log(1-$

$p_n) + p_n n =: \mathrm{lq}_n$ and recall from e.g. the solutions to Sheet 2, p. 4, that $\log(1+x) = \sum_{j\geq 1} \frac{(-1)^{j-1}}{j} x^j$ for every $-1 < x < +1$, hence $n \log(1-p_n) = n(-p_n - \frac{1}{2}p_n^2 - \frac{1}{3}p_n^3 - \frac{1}{4}p_n^4 - \cdots)$, and therefore $\mathrm{lq}_n = -\frac{1}{2}np_n^2 - \frac{1}{3}np_n^3 - \frac{1}{4}np_n^4 - \cdots)$. Obviously,

$$-\tfrac{1}{2} \cdot n \cdot p_n^2 \cdot (\tfrac{1}{1-p_n}) = -\tfrac{1}{2}np_n^2 - \tfrac{1}{2}np_n^3 - \tfrac{1}{2}np_n^4 - \cdots < \mathrm{lq}_n < -\tfrac{1}{2} \cdot n \cdot p_n^2 \quad . \tag{8}$$

The hypothesis about p_n implies that $\frac{1}{1-p_n} \xrightarrow{n\to\infty} 1$ and therefore both $-\frac{1}{2} \cdot n \cdot p_n^2 \cdot (\frac{1}{1-p_n}) \xrightarrow{n\to\infty} 0$ and $-\frac{1}{2} \cdot n \cdot p_n^2 \xrightarrow{n\to\infty} 0$. Hence (8) implies $\mathrm{lq}_n \xrightarrow{n\to\infty} 0$, which completes the proof. □

Lemma 3 (the expected number of isolated vertices w.r.t. the scale $\mathbb{R} \ni c \mapsto \frac{c+\log n}{n} = p_n$)**.** *For every $c \in \mathbb{R}$ we have $\mathsf{Ex}_{\mathrm{G}(n,\frac{c+\log n}{n})}[\mathrm{X}] \xrightarrow{n\to\infty} \exp(-c)$.*

Proof of Lemma 3. We abbreviate $p_{c,n} := \frac{c+\log n}{n}$ and $\mathsf{Ex} := \mathsf{Ex}_{\mathrm{G}(n,p_{c,n})}$. Then $\mathsf{Ex}[\mathrm{X}] = n \cdot (1-p_{c,n})^{n-1}$ $= n(1-p_{c,n})^{-1}(1-p_{c,n})^n \sim$ ($p_{c,n}$ satisfies hypotheses of Lemma 2) $\sim n \cdot (1-p_{c,n})^{-1} \cdot \exp(-p_{c,n} \cdot n)$ $= n \cdot (1-p_{c,n})^{-1} \cdot \exp(-c) \cdot n^{-1} = (1-p_{c,n})^{-1} \cdot \exp(-c) \sim \exp(-c)$. □

We abbreviate $p_n := \frac{c+\log n}{n}$ and $\mathsf{Pr} := \mathsf{Pr}_{\mathrm{G}(n,p)}$.

For every $G \in \mathcal{G}_n$ and every $i \in [n]$ we define $\mathrm{X}_i\colon \mathcal{G}_n \to \mathbb{Z}_{\geq 0}$ by $\mathrm{X}_i(G) := 1$ if i is isolated in G and 0 otherwise. Moreover, we define $\mathrm{X}\colon \mathcal{G}_n \to \mathbb{Z}_{\geq 0}$ as $\mathrm{X} := \sum_{1\leq i\leq n} \mathrm{X}_i$. Finally, for every $0 \leq s \leq n$ we define

$$\xi_s := \sum_{I \in \binom{[n]}{s}} \mathsf{Pr}[\bigvee_{i\in I} (\mathrm{X}_i = 1)] \quad , \tag{9}$$

with the usual understanding that a disjunction over an empty set is the entire sample space, i.e. $\mu_0 = 1$. Now we can calculate

$$\begin{aligned} \mathsf{Pr}[\mathrm{X}=0] &= \mathsf{Pr}[\bigwedge_{i\in[n]} (\mathrm{X}_i = 0)] = 1 - \mathsf{Pr}[\bigvee_{i\in[n]} (\mathrm{X}_i = 1)] \\ \text{(inclusion-exclusion)} \qquad &= 1 - \sum_{1\leq s\leq n} (-1)^{s-1} \cdot \xi_s = \sum_{0\leq s\leq n} (-1)^s \cdot \xi_s \quad , \end{aligned} \tag{10}$$

and by Bonferroni's inequalities we know that for every $b \leq \lfloor \frac{n-1}{2} \rfloor$,

$$\sum_{0\leq s\leq 2b+1} (-1)^s \, \xi_s \leq \mathsf{Pr}[\mathrm{X}=0] \leq \sum_{0\leq s\leq 2b} (-1)^s \, \xi_s \quad . \tag{11}$$

For every $0 \leq s \leq n$ we have $\xi_s = \binom{n}{s} \cdot (1-p_n)^{s(n-s)+\binom{s}{2}}$ and it can be shown that with $\mu := \mathsf{Ex}_{\mathrm{G}(n,p_n)}[\mathrm{X}] = n(1-p_n)^{n-1}$,

$$\forall \varepsilon > 0\colon \exists n_0 > 0\colon \forall n > n_0\colon \forall s \in \{0,1,\ldots,2b\}\colon \quad \left|\xi_s - \tfrac{\mu^s}{s!}\right| \leq \tfrac{\varepsilon}{2(2b+1)} \quad . \tag{12}$$

Moreover, from (12) and (11) combined it follows that

$$\forall \varepsilon > 0\colon \exists n_0 > 0\colon \forall n > n_0\colon \; -\varepsilon + \exp(-\mu) \leq \mathsf{Pr}[\mathrm{X}=0] \leq \varepsilon + \exp(-\mu) \quad , \tag{13}$$

(the term $\exp(-\mu)$ comes from summing the terms $\frac{(-\mu)^s}{s!}$ and estimating a finite sum by a series) and therefore

$$\mathsf{Pr}[\mathrm{X}=0] \xrightarrow{n\to\infty} \exp(-\mu) \quad . \tag{14}$$

By continuity of exp, Lemma 3 and (14) imply the limit in Problem 6.3.