Lec 1, 18.4.12

Notiztitel 17.04.2012

Random Graphs (MA 5208)

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1. Introduction

random graph : (outcome of) a random experiment that produces a graph

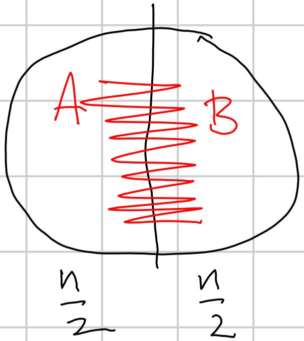
why study random graphs ?

1.1. Probabilistic proof of existence

(historically: first reason)

Thm 1.1 Every graph $G=(V,E)$ contains a bipartite subgraph $H=(V,F)$ with $|F| \geq \frac{|E|}{2}$.

e.g.: $G = K_n$



$|F| = \frac{n^2}{4}$

$|E| = \frac{n(n-1)}{2}$

<u>Proof</u>:

- flip a fair coin for every vertex:
- put all head - vertices into class A
  (Kopf)
  all tails - vertices into class B
  (Zahl)
- Consider each edge $\{x,y\} \in E$ individually:

x	y	Pr	$\{x,y\}$ good for F ?
A	A	$\frac{1}{4}$	
A	B	$\left.\begin{matrix}\frac{1}{4} \\ \frac{1}{4}\end{matrix}\right\} = \frac{1}{2}$	yes
B	A		yes
B	B	$\frac{1}{4}$	

$\Rightarrow$ $Ex[\#\text{ edges between } A \text{ and } B] = \frac{|E|}{2}$

$\Rightarrow$ there must be a flipping which gives <u>at least</u> $\frac{|E|}{2}$ edges <u>between</u> A and B.

1.2 Structure of almost all graphs

$$[n] := \{1, 2, \dots, n\}$$

$$\mathcal{G}_n := \{ \text{ all graphs on vertex set } [n] \}$$

$$|\mathcal{G}_n| = 2^{\binom{n}{2}}.$$

<u>Def:</u> Let $\mathcal{P}_n \subset \mathcal{G}_n$ and $\mathcal{P} := \bigcup_{n=1}^{\infty} \mathcal{P}_n$.

almost all graphs have property $\mathcal{P}$

iff $$\lim_{n \to \infty} \frac{|\mathcal{P}_n|}{|\mathcal{G}_n|} = 1.$$

Thm 1.2 Almost all graphs have no isolated vertex.

Proof: Let $\mathcal{I}_n := \{ G \in \mathcal{G}_n : G \text{ has at least one isolated vertex} \}$

goal: $\frac{|\mathcal{I}_n|}{|\mathcal{G}_n|} \xrightarrow{n \to \infty} 0$.

$$\frac{|\mathcal{I}_n|}{|\mathcal{G}_n|} \leq \frac{n \cdot 2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = n \, 2^{\frac{(n-1)(n-2)}{2} - \frac{n(n-1)}{2}}$$

$$= n \, 2^{\frac{n-1}{2}\left((n-2)-n\right)} = n \, 2^{-(n-1)} = \frac{n}{2^{n-1}} \longrightarrow 0.$$

$\square$

Similarly: almost all graphs are connected, have triangles, Hamilton cycles, ...

$\boxed{\text{„Simpler model": } \quad G(n, \tfrac{1}{2}):}$

vertex set $[n]$, flip a fair coin for every pair $\{i,j\} \in \binom{[n]}{2}$ to decide whether there is an edge
denote the outcome by r.v. $G(n, \tfrac{1}{2})$

for some fixed graph $G \in \mathcal{G}_n$:

$$\Pr\left[G(n, \tfrac{1}{2}) = G \right] = \left(\tfrac{1}{2}\right)^{\binom{n}{2}}$$

$\rightarrow$ can generate a graph from $\mathcal{G}_n$ uniformly at random
zufällig gleichverteilt

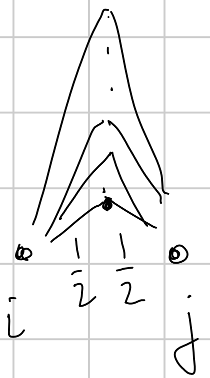
$$\Rightarrow \Pr\left[G(n, \tfrac{1}{2}) \in \mathcal{P}_n \right] = \sum_{G \in \mathcal{P}_n} \Pr\left[G(n, \tfrac{1}{2}) = G \right]$$

$$= |\mathcal{P}_n| \cdot \left(\tfrac{1}{2}\right)^{\binom{n}{2}} = \frac{|\mathcal{P}_n|}{2^{\binom{n}{2}}} = \frac{|\mathcal{P}_n|}{|\mathcal{G}_n|}.$$

Thm 1.3 In almost every graph [every pair of vertices has at least a common neighbour.] $=: P$

Note: Thm 1.3 $\Rightarrow$ Thm 1.2

Proof: For $i,j \in [n]$ let



$$Z_{i,j} := \begin{cases} 0 & \text{othw. (sonst)} \\ 1 & i,j \text{ have no common neighbour} \end{cases}$$

$$\Pr[Z_{i,j} = 1] = \left(\frac{3}{4}\right)^{n-2}$$

$$Z := \sum_{1 \le i < j \le n} Z_{ij}$$

$$\Pr\left[G(n,\tfrac{1}{2}) \notin \mathcal{B}\right] = \Pr[Z \geq 1] \leq \operatorname{Ex}[Z]$$

Markov inequ. : $\Pr[X \geq t] \leq \dfrac{\operatorname{Ex}[X]}{t}$

$$\leq \operatorname{Ex}\Big[\sum_{ij} Z_{ij}\Big]$$

$$= \sum_{ij} \operatorname{Ex}[Z_{ij}] = \sum_{ij} \Pr[Z_{ij} = 1] = \binom{n}{2}\left(\frac{3}{4}\right)^{n-2} \longrightarrow 0.$$

$\square$

1.3 The evolution of $G(n,p)$

$G(n,p) :=$ every edge exists with probability p.

$$\mathrm{Ex}[\#\text{ triangles in } G(n,p)] = \binom{n}{3} p^3 \simeq \frac{n^3 p^3}{6}$$

Thm 1.4

$$\left.\begin{array}{ll} \text{if} & pn \to 0 \\ \text{if} & pn \to c \\ \text{if} & pn \to \infty \end{array}\right. \quad \text{then } \Pr[\exists \text{ triangle in } G(n,p)] \xrightarrow{n\to\infty} \begin{cases} 0 \\ 1 - e^{-\frac{c^3}{6}} \\ 1 \end{cases}$$

Thm 1.5 $L_1(G) :=$ size of largest connected comp. in G

$$
\left.\begin{array}{ll}
\text{if} & pn < 1-\varepsilon \\
\text{if} & pn = 1 \\
\text{if} & pn > 1+\varepsilon
\end{array}\right. \quad \text{then} \quad \Pr\left[\begin{array}{l}
L_1(G(n,p)) = O(\log n) \\
L_1(G(n,p)) = \Theta(n^{2/3}) \\
L_1(G(n,p)) = \Omega(n)
\end{array}\right] \longrightarrow 1
$$

„double jump". at threshold $p = \frac{1}{n}$

Qu: Threshold for other properties ?

1.4. Almost all graphs from a certain class

e.g.: almost all trees ... 5-regular graphs ... planar graphs

(Huge) problem: how to generate these u.a.r.?
→ go back to Counting.

Thm 1.6

Almost all trees have a linear number of leaves.

Open question 1.7

Almost all 5-regular graphs are 3-colourable.

1.5 Probabilistic analysis of algorithms

- „Find optimum colouring of G" is a hard problem
- even no approximation algo with ratio $< n^{1-\varepsilon}$ known.

Thm 1.7

a) $\Pr\left[\chi(G(n, \tfrac{1}{2})) \sim \dfrac{n}{2 \log n} \right] \xrightarrow{n \to \infty} 1$

b) $\Pr$[random greedy algo needs less than $\dfrac{n}{\log n}$ colours] $\xrightarrow{n \to \infty} 1$.

$\leadsto$ 2-Approximation ratio.