Lec.2, 25.4.12

⚠ no lecture next week (Wed., May 2), → Fachschafts VV

1.6 Tools from Probability Theory

Landau notation

Def 1.8 $f, g : \mathbb{N} \longrightarrow \mathbb{R}$

a) $f(n) = O(g(n)) \;:\Leftrightarrow\; \exists c \in \mathbb{R}_{>0} \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 : |f(n)| \leq c \cdot |g(n)|$

b) $f(n) = \Omega(g(n)) \;:\Leftrightarrow\; \ldots \geq \ldots$

c) $f(n) = \Theta(g(n)) \;:\Leftrightarrow\; \exists c_1, c_2 \in \mathbb{R}_{>0} \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 :$

$c_1 |g(n)| \leq |f(n)| \leq c_2 \cdot |g(n)|.$

$\Leftrightarrow \; f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

d) $f(n) = o(g(n)) \;:\Leftrightarrow\; \forall c \in \mathbb{R}_{>0} \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 \; |f(n)| \leq c \cdot |g(n)|$

$\Leftrightarrow \; \lim_{n \to \infty} \frac{|f(n)|}{|g(n)|} = 0$

$\Leftrightarrow: \; f(n) \ll g(n)$

e) $f(n) = \omega(g(n)) \;:\Leftrightarrow\; \forall c \; \ldots \geq \ldots$

$\Leftrightarrow \; \lim_{n \to \infty} \frac{|f(n)|}{|g(n)|} = +\infty$

$\Leftrightarrow: \; f(n) \gg g(n)$

f) $f(n) \sim g(n) \;:\Leftrightarrow\; f(n) = (1 + o(1)) \cdot g(n)$

$\Leftrightarrow \; \lim_{n \to \infty} \frac{|f(n)|}{|g(n)|} = 1$

Example 1.9

a) $7n = O(n)$

b) $n \ll n^{1+\varepsilon}$

c) $n \cancel{\ll} 7n$

d) $(\log n)^7 \ll n^{\frac{1}{17}}$

e) $n^2 + 17n = \Theta(n^2)$

f) $\frac{n^2}{\log\log n} \neq O(n^{1.9})$

g) $\binom{n}{k} \sim \frac{n^k}{k!}$

$k=2$: $\frac{\frac{n^2-n}{2}}{n^2/2} = 1 - \frac{1}{n} \to 1$

First moment

Thm 1.10 (Markov-inequality)

Let X be a non-negative random variable and $t \in \mathbb{R}$, $t > 0$. Then

$$\Pr[X \geq t] \leq \frac{\mathrm{Ex}[X]}{t}.$$

In particular, $\Pr[X \geq 1] \leq \mathrm{Ex}[X]$. □

Application:

Thm 1.4 (revisited)

Consider random graph $G(n,p)$ with some probability $p \ll \frac{1}{n}$. Let $X :=$ number of triangles in $G(n,p)$.

Then

$$\Pr[\exists \text{ triangle in } G(n,p)] = \Pr[X \geq 1] \overset{1.10}{\leq} \mathrm{Ex}[X] \xrightarrow[n\to\infty]{} 0$$

!

Proof:

For a triple $1 \leq a < b < c \leq n$ denote by

$$X_{a,b,c} := \begin{cases} 1 & \underbrace{G(n,p)[\{a,b,c\}] \cong K_3}_{\{a,b,c\} \text{ forms a triangle in } G(n,p)} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \Pr[X_{a,b,c} = 1] = p^3$$

and $\sum_{1 \leq a < b < c \leq n} X_{a,b,c} = X$

$$\Rightarrow \operatorname{Ex}[X] = \operatorname{Ex}\Big[\sum_{a<b<c} X_{a,b,c}\Big]$$

$$= \sum_{a<b<c} \operatorname{Ex}[X_{a,b,c}]$$

$$= \sum_{a<b<c} \Pr[X_{a,b,c} = 1]$$

$$\left\{ \begin{aligned} \operatorname{Ex}[Y] &= 0 \cdot \Pr[Y=0] \\ &+ 1 \cdot \Pr[Y=1] \\ &+ 2 \cdot \Pr[Y=2] \\ &+ \ldots \\ &= \Pr[Y=1] \end{aligned} \right.$$

$$= \sum_{a<b<c} p^3$$

$$= \binom{n}{3} p^3 \sim \frac{n^3}{6} p^3 \ll \frac{n^3}{6}\left(\frac{1}{n}\right)^3 = \frac{1}{6}$$

$$\Rightarrow \lim_{n \to \infty} \frac{\operatorname{Ex}[X]}{\frac{1}{6}} = 0$$

$\square$

Recall: $\Pr[X \geq 1] \leq \operatorname{Ex}[X] \xrightarrow{n \to \infty} 0$

But observe: „$\operatorname{Ex}[X]$ large" does not $\underbrace{\ }_{\text{(always)}}$ imply $\Pr[X \geq 1] \to 1$.

Second moment

random variable Y, $\mu := \mathrm{Ex}[Y]$.

$$\begin{aligned}\mathrm{Var}[Y] &:= \mathrm{Ex}[(Y-\mu)^2] \\ &= \mathrm{Ex}[Y^2 - 2\mu Y + \mu^2] \\ &= \mathrm{Ex}[Y^2] - 2\mu\,\mathrm{Ex}[Y] + \mu^2 \\ &= \mathrm{Ex}[Y^2] - \mu^2 .\end{aligned}$$

Thm 1.11 (Chebychev-inequality)

Let Y be a non-negative random variable and $s \in \mathbb{R}$, $s > 0$. Then

a) $$\Pr[\,|Y-\mu| \geq s\,] \leq \frac{\mathrm{Var}[Y]}{s^2}.$$

b) In particular, $$\Pr[Y=0] \leq \frac{\mathrm{Ex}[Y^2]}{\mu^2} - 1.$$

Proof: a) apply Thm 1.10 with $X := (Y-\mu)^2$, $t := s^2$.

b) Set $s := \mu$ in a) and obtain

$$\Pr[Y=0] \leq \Pr[\,|Y-\mu| \geq \mu\,] \leq \frac{\mathrm{Var}[Y]}{\mu^2} = \frac{\mathrm{Ex}[Y^2]}{\mu^2} - 1.$$

$\square$

Lemma 1.12

Let $Y = \sum_{i \in I} X_i$, $X_i \in \{0,1\}$, $\mu = \mathrm{Ex}[Y]$.

For $i \neq j \in I$ we write $i \sim j$ iff the events $X_i = 1$ and $X_j = 1$ are independent. Set

$$\Delta := \sum_{(i,j):\, i \neq j} \Pr[(X_i = 1) \wedge (X_j = 1)]$$

$$\tilde{\Delta} := \sum_{(i,j):\, i \sim j} \Pr[(X_i = 1) \wedge (X_j = 1)].$$ Then

a) $\mathrm{Ex}[Y^2] = \mu + \Delta$

b) $\Delta \leq \mu^2 + \tilde{\Delta}$

c) $\Pr[Y = 0] \leq \frac{1}{\mu} + \frac{\Delta}{\mu^2} - 1$

d) $\Pr[Y = 0] \leq \frac{1}{\mu} + \frac{\tilde{\Delta}}{\mu^2}$.

Proof:

a) $$\mathrm{Ex}[Y^2] = \mathrm{Ex}\left[\left(\sum_i X_i\right)^2\right]$$

$$= \sum_i \mathrm{Ex}[X_i^2] + \sum_{i \neq j} \mathrm{Ex}[X_i X_j]$$

$$= \sum_i \mathrm{Ex}[X_i] + \sum_{i \neq j} \Pr[(X_i = 1) \wedge (X_j = 1)]$$

$$= \mu + \Delta.$$

b) $$\Delta = \sum_{i \not\sim j} \Pr[(X_i = 1) \wedge (X_j = 1)] + \sum_{i \sim j} \Pr[(X_i = 1) \wedge (X_j = 1)]$$

$$= \sum_{i \not\sim j} \Pr[X_i = 1] \cdot \Pr[X_j = 1] + \tilde{\Delta}$$

$$\le \Big(\sum_{i \in I} \Pr[X_i = 1]\Big)^2 + \tilde{\Delta}$$

$$= \mu^2 + \tilde{\Delta}$$

c) $$\Pr[Y = 0] \underset{1.11b)}{\le} \frac{E[Y^2]}{\mu^2} - 1 \underset{a)}{=} \frac{\mu + \Delta}{\mu^2} - 1$$

$$= \frac{1}{\mu} + \frac{\Delta}{\mu^2} - 1$$

d) $$\Pr[Y = 0] \underset{c)}{\le} \frac{1}{\mu} + \frac{\Delta}{\mu^2} - 1 \underset{b)}{\le} \frac{1}{\mu} + \frac{\mu^2 + \tilde{\Delta}}{\mu^2} - 1$$

$$= \frac{1}{\mu} + \frac{\tilde{\Delta}}{\mu^2}.$$

Applications

Prop. 1.13 Let $\alpha : \mathbb{N} \to \mathbb{R}$ be an arbitrary function, let Y denote the number of edges in $G(n, \frac{1}{2})$. Then

$$\Pr\Big[\,\Big|Y - \tfrac{1}{2}\binom{n}{2}\Big| \ge \alpha(n) \cdot n\,\Big] \le \frac{1}{4\alpha(n)^2}$$

Proof:

$Y :=$ number of edges in $G(n, \frac{1}{2})$

$X_{u,v} := \begin{cases} 1 & \{u,v\} \text{ is an edge in } G(n,\frac{1}{2}) \\ 0 & \end{cases}$

$\Rightarrow Y = \sum_{i<j} X_{u,v}$, $Pr[X_{u,v} = 1] = \frac{1}{2}$

$\mu = \frac{1}{2}\binom{n}{2}$, $\tilde{\Delta} = 0$

$\Rightarrow Ex[Y^2] \underset{1.12a)}{=} \mu + \Delta \underset{1.12b)}{\leq} \mu + \mu^2 + \tilde{\Delta} = \mu + \mu^2$

$\Rightarrow Pr[\,|Y-\mu| \geq \underbrace{\alpha(n) \cdot n}_{=:s}\,] \underset{1.11a)}{\leq} \frac{Var[Y]}{\alpha(n)^2 n^2}$

$= \frac{Ex[Y^2] - \mu^2}{\alpha(n)^2 n^2} \leq \frac{\mu}{\alpha(n)^2 n^2} = \frac{\frac{1}{2}\binom{n}{2}}{\alpha(n)^2 n^2} \leq \frac{1}{4\alpha(n)^2}$

$\to 0$.

Thm 1.4. (re-revisited)

Let $p \gg \frac{1}{n}$ and $X :=$ number of triangles in $G(n,p)$. Then $Pr[X=0] \xrightarrow{n\to\infty} 0$.

Proof:

$X_{a,b,c}$ as in Proof of Thm 1.4

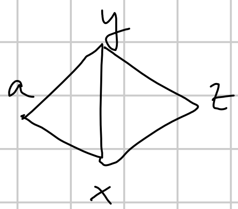
$Pr[X_{a,b,c} = 1] = p^3$

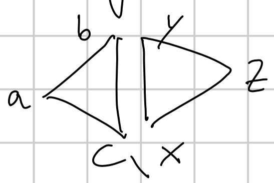
$$\sum_{a<b<c} X_{a,b,c} = X$$

$$\mu = Ex[X] = \binom{n}{3} p^3 \sim \frac{n^3}{6} p^3$$

$$\tilde{\Delta} = ?$$

$$= \sum_{a,b,c \sim x,y,z} Pr\left[\left(X_{a,b,c} = 1\right) \wedge \left(X_{x,y,z} = 1\right) \right]$$

$$= \sum_{x,y,a,z :} Pr\Big[$$  formed in $G(n,p)\Big]$



$$\leq n^4 p^5$$

1.12d)

$$\Rightarrow Pr[X=0] \leq \frac{1}{\mu} + \frac{\tilde{\Delta}}{\mu^2}$$

$$\sim \frac{1}{\mu} + \frac{\tilde{\Delta} \; 36}{n^6 p^6} \leq \frac{1}{\mu} + \frac{n^4 p^5 \; 36}{n^6 p^6}$$

$$\sim \frac{6}{(np)^3} + \frac{36}{n \, np} \longrightarrow 0.$$

Since $p >> \frac{1}{n}$

$\frac{p}{\frac{1}{n}} = p \cdot n \longrightarrow \infty$, thus $\longrightarrow \infty$, thus

$\square$