# 3. Evolution of random graphs

Lecture 20.6.12

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recall Thm 1.4:

if $p \ll \frac{1}{n}$, then $\Pr[\,G(n,p) \text{ has a triangle}\,] \xrightarrow{n\to\infty} 0$

if $p \gg \frac{1}{n}$, then $\Pr[\,G(n,p) \text{ has a triangle}\,] \xrightarrow{n\to\infty} 1$.

## Definition 3.1

Let $\mathcal{P}$ be a graph property (a class of graph closed under isomorphism).

a) A function $t(n)$ is called a threshold function for the property $\mathcal{P}$, if

$p(n) \ll t(n) \Rightarrow \Pr[\,G(n,p) \in \mathcal{P}\,] \xrightarrow{n\to\infty} 0$ „0-statement"

$p(n) \gg t(n) \Rightarrow \Pr[\qquad\qquad] \xrightarrow{n\to\infty} 1$. „1-statement"

b) If one replaces $p(n) \ll t(n)$ by $\forall \varepsilon > 0 \quad p(n) \leq (1-\varepsilon)\, t(n)$

$p(n) \gg t(n)$ by $\forall \varepsilon > 0 \quad p(n) \geq (1+\varepsilon)\, t(n)$

and still have the same implications, then $t(n)$ is called a sharp threshold.

Theorem 3.2

a) $t(n) = \frac{\ln n}{n}$ is a sharp threshold for " G has no isolated vertices "

b) $t(n) = \frac{\ln n}{n}$ is a (sharp) thresh d for " G is connected "

c) $\forall\, 0 < x < \frac{1}{2}$ : $e^{-x-x^2} \leq 1 - x \leq e^{-x}$

d) For $x \to 0$ $\quad 1 - x \sim e^{-x}$

Proof: c) Calculus

d) $1 \leftarrow e^{-x^2} = \frac{e^{-x-x^2}}{e^{-x}} \leq \frac{1-x}{e^{-x}} \overset{c)}{\leq} 1$

a) $v \in [n]$, $X_v := \begin{cases} 1 & \deg(v) = 0 \\ 0 & \text{otherw} \end{cases}$

$$Y := \#\text{ isol. verts} = \sum_{v \in [n]} X_v$$

$$\mu := \mathrm{Ex}[Y] = \sum_{v \in [n]} \mathrm{Ex}[X_v] = \sum_{v \in [n]} \Pr[X_v = 1] = n \cdot (1-p)^{n-1}$$

0-Statement: need to show: $\forall \varepsilon > 0$: If $\boxed{p \le (1-\varepsilon) \frac{\ln(n)}{n}}$, then

$$\Pr[Y = 0] = \Pr[G(n,p) \text{ has no isolated vertices}] \xrightarrow{n \to \infty} 0$$

$$\mu = \frac{n}{1-p} (1-p)^n \overset{\text{d}}{\sim} \frac{n}{1-p} e^{-pn} \ge \frac{n}{1-p} e^{-(1-\varepsilon)\ln(n)}$$

$$= \frac{n}{1-p} n^{-(1-\varepsilon)} = \frac{n^{\varepsilon}}{1-p} \xrightarrow{n \to \infty} \infty$$

$$\Delta := \sum_{u \ne w} \Pr[(X_u = 1) \wedge (X_w = 1)] = \sum_{u \ne w} \Pr[X_u = 1] \cdot \Pr[X_w = 1 \mid X_u = 1]$$

$$= \sum_{u \neq w} (1-p)^{n-1} \cdot (1-p)^{n-2} \le n^2 (1-p)^{2n-3}$$


$$\frac{\Delta}{\mu^2} \le \frac{n^2 (1-p)^{2n-3}}{\left(n (1-p)^{n-1}\right)^2} = \frac{n^2 (1-p)^{2n-3}}{n^2 (1-p)^{2n-2}} = \frac{1}{1-p} \xrightarrow{n\to\infty} 1$$

$$\Rightarrow \Pr[Y=0] \underset{1.12\,d)}{\le} \underset{\to 0}{\frac{1}{\mu}} + \underset{\to 1}{\frac{\Delta}{\mu^2}} - 1 \xrightarrow{n\to\infty} 0.$$

<u>1-statement</u> need to show: $\forall \varepsilon > 0$ if $\boxed{p \ge (1+\varepsilon)\frac{\ln(n)}{n}}$, then

$$\Pr[Y=0] = \Pr[G(n,p) \text{ has no isolated vertices}] \xrightarrow{n\to\infty} 1$$

$$\Pr[Y=0] = 1 - \Pr[Y \ge 1]$$

$$\mu = n(1-p)^{n-1} = \frac{n}{1-p}(1-p)^n \overset{c)}{\leq} \frac{n}{1-p} e^{-pn} \leq \frac{n}{1-p} e^{-(1+\varepsilon)\ln n}$$

$$= \frac{n}{1-p} n^{-(1+\varepsilon)} = \frac{1}{(1-p)\, n^{\varepsilon}} \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \quad \Pr[Y \geq 1] \leq \mathrm{Ex}[Y] \longrightarrow 0 .$$

b) Is $G(n,p)$ connected ?

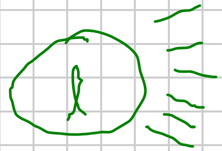
0 - statement: For $p \leq (1-\varepsilon)\frac{\ln(n)}{n}$ we know

$$\Pr[\,G(n,p) \text{ connected}\,] \leq \Pr[\,G(n,p) \text{ has no isolated vertices}\,] \xrightarrow{a)} 0 .$$

1 - statement For $p \geq 6 \frac{\ln(n)}{n}$, we want $\Pr[G(n,p) \text{ connected}] \to 1$.

$$\overset{\shortparallel}{1 - \Pr[G(n,p) \text{ not connected}]}$$

$X_\ell :=$ # connected components of order exactly $\ell$ in $G(n,p)$, for $1 \le \ell \le \frac{n}{2}$.




$\ell \ln(n) -$

$$\mathrm{Ex}[X_\ell] \le \underbrace{n^{\ell}}_{\#\text{ possible comp}} \underbrace{(1-p)^{\ell(n-\ell)}}_{\text{no edges to exterior}}$$

$$\le n^{\ell} e^{-p\ell(n-\ell)} = e^{\ell \ln(n) - p\ell(n-\ell)}$$

$$\le e^{\ell \ln(n) - 6\frac{\ln(n)}{n}\ell\left(n-\frac{n}{2}\right)}$$

$$= e^{\ell \ln(n) - 3\ell\ln(n)} = e^{-2\ell\ln(n)} = n^{-2\ell} \le \frac{1}{n^2}$$

$$\Rightarrow \sum_{\ell=1}^{n/2} \mathrm{Ex}[X_\ell] \le \frac{n}{2}\cdot\frac{1}{n^2} = \frac{1}{2n} \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \Pr[G(n,p) \text{ not connected}] = \Pr\left[\sum_{\ell=1}^{\frac{n}{2}} X_\ell \ge 1\right] \le \mathrm{Ex}\left[\sum_{\ell=1}^{\frac{n}{2}} X_\ell\right]$$

$$= \sum \mathrm{Ex}[X_\ell] \longrightarrow 0.$$

qed

## Exercise 3.3

Find a function $p = p(n)$ such that for $X :=$ # spanning trees in $G(n,p)$

$Ex[X] \xrightarrow{n\to\infty} \infty$

$Pr[X \geq 1] \xrightarrow{n\to\infty} 0$.

## Theorem 3.4

Let $\rho(H) := \frac{|E(H)|}{|V(H)|}$ and call a graph balanced, iff

$\rho(H) \geq \rho(H')$ for all subgraphs $H' \subsetneq H$.

Then for a balanced graph $H$ a threshold

for the existence of a copy of $H$ in $G(n,p)$ is $t(n) = n^{-1/\rho(H)}$.

**Proof:** Let $v := |V(H)|$ and $e := |E(H)|$. Now $t = n^{-v/e}$

Note: $H, v, e$ are fixed, i.e. do not grow with $n$.

**0-Statement:** $p \ll n^{-v/e}$. Then $p \cdot n^{v/e} = \frac{p}{n^{-v/e}} \to 0$

$Y :=$ # copies of $H$ in $G(n,p)$

$$\Pr[H \subset G(n,p)] = \Pr[Y \geq 1] \overset{1.10}{\leq} Ex[Y] = \Theta(n^v p^e)$$

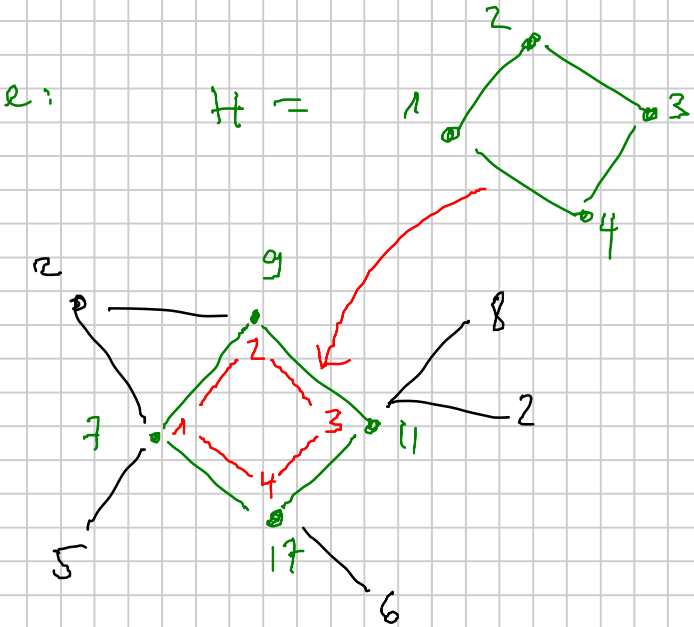
$$= \Theta\left( \left(n^{\frac{v}{e}} p\right)^e \right) \longrightarrow 0.$$

**1-Statement:** $p \gg n^{-v/e}$ Then $p \cdot n^{v/e} = \frac{p}{n^{-v/e}} \to \infty$

Let $S \subset [n]$ be a set of size $v$ and denote by $A_S$ the event that $G(n,p)[S]$ contains the „lexicographic first copy" of $H$.

Example:

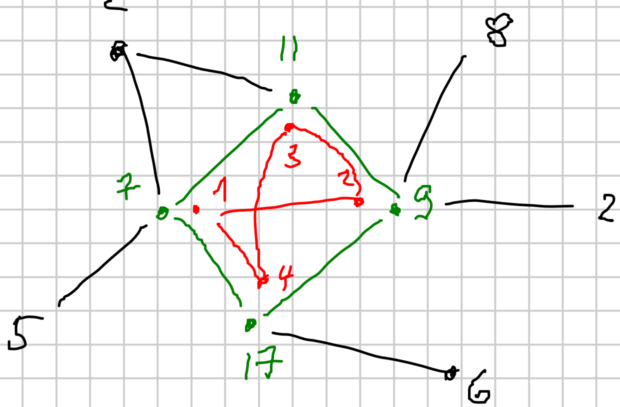
$H =$




$G_1 =$

$G_1[\{7, 9, 11, 17\}]$ contains a „lexicographic first" copy of $H$

$G_2 =$




$G_2[\{7, 9, 11, 17\}]$ does not contain a lexicographic first copy of $H$

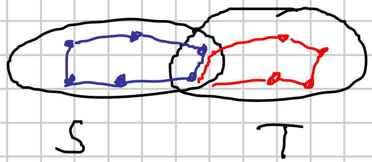
Set $X_S := \begin{cases} 1 & A_S \\ 0 & \text{othw.} \end{cases}$

$$Y := \sum_{S \in \binom{[n]}{v}} X_S \leq \# \text{ copies of } H \text{ in } G(n,p)$$

$$\mu := \mathrm{Ex}[Y] = \sum_{S \in \binom{[n]}{v}} \mathrm{Ex}[X_S] = \Theta(n^v p^e) = \Theta\left( \left(n^{\frac{v}{e}} p\right)^e \right) \to \infty.$$

Consider $S, T \in \binom{[n]}{v}$. The events $A_S$ and $A_T$ are independent if $|S \cap T| \leq 1$. We have to bound

$$\tilde{\Delta} := \sum_{S,T \,:\, A_S, A_T \text{ dependent}} \Pr[X_S = 1 \wedge X_T = 1] = \sum_{v'=2}^{v-1} \sum_{S,T:\, |S \cap T| = v'} \Pr[A_S \wedge A_T]$$




Suppose $A_S \wedge A_T$ holds.

Then there are two fixed copies $H_1$ and $H_2$ of $H$ in $G(n,p)[S \cup T]$. Denote the graph obtained by the union of $H_1$ and $H_2$ by $\bar{H}$, and the graph obtained by their intersection by $H'$.

Then $v' = |V(H')|$ and we let $e' := |E(H')|$.

Observe that $H' \subset H$, hence $\frac{e'}{v'} = \rho(H') \leq \rho(H) = \frac{e}{v}$,

thus $e' \leq v' \cdot \frac{e}{v}$.

Also, $|V(\bar{H})| = 2v - v'$ and $|E(\bar{H})| = 2e - e'$,

hence $\Pr[A_S \wedge A_T] = p^{2e-e'} \leq p^{2e - v'\frac{e}{v}}$.

Summing up we obtain:

$$\frac{\tilde{\Delta}}{\mu^2} \leq \sum_{v'=2}^{v-1} \sum_{S,T:\, |S\cap T| = v'} \frac{p^{2e - v'\frac{e}{v}}}{\mu^2} = \sum_{v'=2}^{v-1} O\left( \frac{\binom{n}{v}\binom{v}{v'}\binom{n-v}{v-v'} \; p^{2e - v'\frac{e}{v}}}{n^{2v} \; p^{2e}} \right)$$

$$= \sum_{v'=2}^{v-1} O\left( \frac{n^{v} \, n^{v-v'} \; p^{2e - v'\frac{e}{v}}}{n^{2v} \, p^{2e}} \right) = \sum_{v'=2}^{v-1} O\left( \frac{1}{n^{v'} \; p^{v'\frac{e}{v}}} \right)$$

$$= \sum_{v'=2}^{v-1} O\left( \left( \frac{1}{n^{\frac{v}{e}} p} \right)^{v'\frac{e}{v}} \right) \longrightarrow 0 .$$

This gives

$$\Pr[\text{no copies of } H] \leq \Pr[Y = 0] \leq \frac{1}{\mu} + \frac{\tilde{\Delta}}{\mu^2} \longrightarrow 0 .$$

qed