RANDOM GRAPHS. LECTURE 10. 27 JUNE 2012. TUM.

Today we look inside the phase transition for connectedness of the random graph $\mathrm{G}(n, p_n)$. Most of today's lecture consists of an exposition of a recent preprint of M. Krivelevich and B. Sudakov (arXiv:1201.6529v2).

We know from the lecture of 20 june 2012 that

$$\mathsf{Pr}_{\mathrm{G(n,p_n)}}[\{G \in \mathcal{G}_n \colon G \text{ is connected}\}] \xrightarrow{n\to\infty} \begin{cases} 0, & \text{if } \frac{p_n}{(\log n)n^{-1}} \xrightarrow{n\to\infty} 0 \quad , \\ 1, & \text{if } \frac{p_n}{(\log n)n^{-1}} \xrightarrow{n\to\infty} \infty \quad . \end{cases} \tag{1}$$

Today we 'slow down' and study the a.a.s. sizes of connected components when p_n 'varies on the scale' $c \mapsto c \cdot n^{-1} = p_n$ instead of the much cruder scale $a_n \mapsto a_n \cdot (\log n) \cdot n^{-1} = p_n$ with a_n a sequence with either $a_n \xrightarrow{n\to\infty} 0$ or $a_n \xrightarrow{n\to\infty} \infty$.

Mathematicians have 'zoomed in' on the phase transition even more and found that the scale $c \mapsto n^{-1}+c\cdot n^{-4/3}$ results in a particularly noteworthy a.a.s.-behaviour of the connected components. (Nothing more on this today.)

Throughout these lecture notes we simply ignore the rounding of non-integers to the next integer. It is an exercise to justify this either by 'continuity arguments' or by inserting floors and ceilings at appropriate places.

We start with a technical tool:

Lemma 3.5. Let $0 < \varepsilon < 1$. Let $(b_n) \in \mathbb{N}^{\mathbb{N}}$ with $b_n \in \Theta(n^2)$. Let $(p_n) \in \mathbb{R}^{\mathbb{N}}_{>0}$. For every n let $(X_i)_{1\le i\le b_n}$ denote independent Bernoulli(p_n)-distributed random variables with $\mathrm{im}(X_i) \in \{0,1\}$. Then

(3.5.1) If $p_n = \frac{1-\varepsilon}{n}$, then with $n \to \infty$ we a.a.s. have

$$\left| |\{i \in I \colon X_i = 1\}| - \tfrac{(1-\varepsilon)(4+\varepsilon)}{\varepsilon^2} \log n \right| \le \tfrac{4+\varepsilon}{\varepsilon} \log n \quad , \tag{2}$$

simultaneously for every *interval* $I \subseteq [b_n]$ with $|I| = \frac{4+\varepsilon}{\varepsilon^2} n \log n$.

(3.5.2) If $p_n = \frac{1+\varepsilon}{n}$, then with $n \to \infty$ we a.a.s. have

$$\left| |\{i \in I \colon X_i = 1\}| - \tfrac{1}{2}\varepsilon(1+\varepsilon)n \right| \le n^{\frac{1+\varepsilon}{2}} \quad , \tag{3}$$

simultaneously for every *interval* $I \subseteq [b_n]$ with $|I| = \frac{1}{2}\varepsilon n^2$.

Proof of Lemma 3.5. As to (3.5.1), for every interval $I \subseteq [b_n]$ with $\ell_n := |I| = \frac{4+\varepsilon}{\varepsilon^2} n \log n$ we denote by A_I the event that $\#_I := |\{i \in I \colon X_i = 1\}|$ does *not* satisfy (2). Then

$$\mathsf{Pr}[\bigcup A_I] \le \sum \mathsf{Pr}[A_I] = (b_n - \ell_n + 1) \cdot \mathsf{Pr}[A_{[\ell_n]}] \quad , \tag{4}$$

where union and sum range over all such intervals. Since $\#_{[\ell_n]}$ is Binom(ℓ_n, p_n)-distributed, we have $\mathsf{E}[\#_{[\ell_n]}] = \frac{(1-\varepsilon)(4+\varepsilon)}{\varepsilon^2} \log n$, so (2) has a form which makes it look amenable to Chebyshev's inequality. However, $\mathsf{Var}[\#_{[\ell_n]}] = \ell_n p_n (1-p_n) = \frac{(1-\varepsilon)(4+\varepsilon)}{\varepsilon^2} \log n - \mathcal{O}(n^{-1})$ and therefore Chebyshew merely gives $\mathsf{Pr}[A_{[\ell_n]}] \le C_\varepsilon \cdot (\log n)^{-1} - \mathcal{O}(\frac{1}{n(\log n)^2})$ which because of $b_n \in \Theta(n^2)$ is not enough to imply that $(b_n - \ell_n + 1)\,\mathsf{Pr}[A_{[\ell_n]}] \xrightarrow{n\to\infty} 0$. But there exist more sophisticated concentration inequalities, for example:

Lemma 3.6. (a Chernoff bound) **.** Let $s \in \mathbb{N}$, let X denote a binom(s,p)-distributed random variable. Then for every $\delta \geq 0$,

$$\mathsf{Pr}[|X - \mathsf{E}[X]| \geq \delta] \leq \exp(-\tfrac{\delta^2}{2\cdot\mathsf{E}[X]}) \quad . \tag{5}$$

(For a proof of Lemma 3.6 see e.g. the solutions to Problem 6.2 in the exercises for this course.)

In our application, with $\delta := \frac{4+\varepsilon}{\varepsilon}\log n$ and $X := \#_{[\ell_n]}$ and $\mathsf{E}[X] = \frac{(1-\varepsilon)(4+\varepsilon)}{\varepsilon^2}\log n$ Lemma 3.6 gives

$$\begin{aligned}\mathsf{Pr}[A_{[\ell_n]}] &\leq \exp(-\frac{\frac{(4+\varepsilon)^2}{\varepsilon^2}(\log n)^2}{2\frac{(1-\varepsilon)(4+\varepsilon)}{\varepsilon^2}\log n}) \\ &= \exp(-\tfrac{1}{2}\ \tfrac{4+\varepsilon}{1-\varepsilon}\ \log n) = n^{-\frac{2+\frac{1}{2}\varepsilon}{1-\varepsilon}} \in o_{n\to\infty}(n^{-2}) \quad ,\end{aligned} \tag{6}$$

and therefore $(b_n - \ell_n + 1)\ \mathsf{Pr}[A_{[\ell_n]}]$, since in particular $b_n \in O(n^2)$. This proves (3.5.1).

As to (3.5.2), this can be proved by reasoning in the same way as for (3.5.1): defining A_I analogously, with $\delta := n^{\frac{1+\varepsilon}{2}}$ Lemma 3.6 now gives

$$\mathsf{Pr}[A_{[\ell_n]}] \leq \exp(-\frac{n^{1+\varepsilon}}{\varepsilon(1+\varepsilon)n}) = \exp(-\tfrac{1}{\varepsilon(1+\varepsilon)}\cdot n^{\varepsilon}) \quad . \tag{7}$$

Because of $b_n \in O(n^2)$ the bound (7) implies $(b_n - \ell_n + 1)\cdot \mathsf{Pr}[A_{[\ell_n]}] \xrightarrow{n\to\infty} 0$, proving (3.5.2).

We now come to the main statement of today's lecture:

Theorem 3.7(essentially known to Erdős and Rényi in the 1950s; statement about path is a corollary of a result of Ajtai, Komlós and Szemerédi published in 1981)**.** Let $\varepsilon_0 > 0$ be small enough[a] so that for every $0 < \varepsilon < \varepsilon_0$,

(Prop.ε_0.(1) $1 - \frac{1}{2}\varepsilon - \frac{6}{5}\varepsilon^2 > \frac{1}{2}$,

(Prop.ε_0.(2) $\frac{1}{4}\varepsilon^2 - 2\varepsilon^3 - \varepsilon^4 > 0$,

(Prop.ε_0.(3) $\frac{1}{20}\varepsilon^2 - \frac{3}{4}\varepsilon^2 - \frac{9}{25}\varepsilon^4 > \frac{1}{30}\varepsilon^2$.

Then for every every $0 < \varepsilon < \varepsilon_0$,

(1) If $p_n = \frac{1-\varepsilon}{n}$, then

$$\mathsf{Pr}_{\mathrm{G}(n,p_n)}[\{G \in \mathcal{G}_n \colon \text{every connected component of } G \text{ has} \leq \tfrac{4+\varepsilon}{\varepsilon^2}\log n \text{ vertices}\}] \xrightarrow{n\to\infty} 1 \quad . \tag{8}$$

(2) If $p_n = \frac{1+\varepsilon}{n}$, then

$$\mathsf{Pr}_{\mathrm{G}(n,p_n)}[\{G \in \mathcal{G}_n \colon \text{in } G \text{ there is a path of length} \geq \tfrac{1}{5}\varepsilon^2 n\}] \xrightarrow{n\to\infty} 1 \quad . \tag{9}$$

[a]These conditions are functions of many contingent decisions and there is no necessity or absolute meaning to them. We spell them out only because we know that these particular conditions are sufficient and we see no reason to withhold that information. You may ignore these conditions and just think 'for a sufficiently small absolute constant $\varepsilon_0 > 0$'.

Note that Theorem 3.7.2 in particular says that a.a.s. there is a connected component of $\Omega(n)$ vertices, so a 'linear change' from $p_n = \frac{1-\varepsilon}{n}$ to $p_n = \frac{1+\varepsilon}{n}$ almost surely brings about an 'exponential change' from $O(\log n)$ to $\Omega(n)$ in the size of the largest connected component.

It can also be shown (nothing more on that in the present lecture) that for $p_n = \frac{1+\varepsilon}{n}$ we have

$$\mathsf{Pr}_{\mathrm{G}(n,p_n)}[\{G \in \mathcal{G}_n \colon \text{in } G \text{ there is a connected component with} \geq \tfrac{1}{2}\varepsilon n \text{ vertices}\}] \xrightarrow{n\to\infty} 1 \quad . \tag{10}$$

Lemma 3.8. Let H be a graph with vertex set $\mathrm{V}(H) \subseteq [n]$. Then the following algorithm (which is a standard depth-first-search with arbitrary choices made by taking minima), when run with input H, after a finite number of steps has computed sequences[1]

$$(\mathrm{U}_i^{(H)})_{0\le i\le |H|},\ (\mathrm{P}_i^{(H)})_{0\le i\le |H|},\ (\mathrm{F}_i^{(H)})_{0\le i\le |H|} \text{ and } (\mathrm{S}_t^{(H)})_{1\le t\le \binom{|H|}{2}},$$

with each $\mathrm{U}_i^{(H)}$ and $\mathrm{F}_i^{(H)}$ a subset of $\mathrm{V}(H)$, each $\mathrm{P}_i^{(H)}$ an ordered *sequence* of elements of $\mathrm{V}(H)$ and each $\mathrm{S}_t^{(H)}$ a 2-element subset of $\mathrm{V}(H)$, such that all the claims made further below are true:

$\mathrm{U}_0^{(H)} \leftarrow \mathrm{V}(H)\setminus\{\min \mathrm{V}(H)\}$; $\mathrm{P}_0^{(H)} \leftarrow (\min \mathrm{V}(H))$; $\mathrm{F}_0^{(H)} \leftarrow \emptyset$;
$i \leftarrow 0$;
$t \leftarrow 0$;

(A.0) if $\mathrm{F}_i^{(H)} = \mathrm{V}(H)$ goto (A.2)
if $\mathrm{P}_i^{(H)} \neq ()$ then $v \leftarrow \mathrm{last}(\mathrm{P}_i^{(H)})$;
else $v \leftarrow \min\ \mathrm{U}_i^{(H)}$; This is never a minimum over an empty set since we have the invariant $\mathrm{V}(H) = \mathrm{U}_i^{(H)} \cup \mathrm{P}_i^{(H)} \cup \mathrm{F}_i^{(H)}$ (a disjoint union) and, whenever this command is reached, we know $\mathrm{P}_i^{(H)} = \emptyset$ and $\mathrm{F}_i^{(H)} \neq \mathrm{V}(H)$.

(A.1) for $\tilde{v} \in \mathrm{U}_i^{(H)}$ do (moving in the natural order on $\mathrm{U}_i^{(H)}$):

if $\{v,\tilde{v}\} \notin \mathrm{E}(H)$ do

if $\{v,\tilde{v}\} \notin \{\mathrm{S}_1^{(H)},\dots,\mathrm{S}_t^{(H)}\}$ do

$t \leftarrow t+1$;
$\mathrm{S}_t^{(H)} \leftarrow \{v,\tilde{v}\}$;

else do nothing;

else do at this point $\{v,\tilde{v}\}$ is an edge of H.

$t \leftarrow t+1$;
$\mathrm{S}_t^{(H)} \leftarrow \{v,\tilde{v}\}$;
$i \leftarrow i+1$;
(∗) $\mathrm{U}_i^{(H)} \leftarrow \mathrm{U}_{i-1}^{(H)}\setminus\{\tilde{v}\}$; $\mathrm{P}_i^{(H)} \leftarrow \mathrm{append}(\mathrm{P}_{i-1}^{(H)},\tilde{v})$; $\mathrm{F}_i^{(H)} \leftarrow \mathrm{F}_{i-1}^{(H)}$;
break the current for-loop and go to (A.0)

end of for-loop
$i \leftarrow i+1$;
(∗∗) $\mathrm{U}_i^{(H)} \leftarrow \mathrm{U}_{i-1}^{(H)}$; $\mathrm{P}_i^{(H)} \leftarrow \mathrm{P}_{i-1}^{(H)}\setminus\{v\}$; $\mathrm{F}_i^{(H)} \leftarrow \mathrm{F}_{i-1}^{(H)}\cup\{v\}$;
go to (A.0);

(A.2) for each $t \le \tilde{t} < \binom{|H|}{2}$ define $\mathrm{S}_{\tilde{t}}^{(H)}$ to the $\tilde{t}$-th element of $\binom{\mathrm{V}(H)}{2}\setminus\{\mathrm{S}_1^{(H)},\dots,\mathrm{S}_t^{(H)}\}$ with respect to the lexicographic ordering on $\binom{\mathrm{V}(H)}{2}$.

[1] You may like to think of them as '**U**nvisited', 'in **P**rogress', '**F**inished' and '2-**S**et'.

Now for every $1 \leq \tau \leq \binom{|H|}{2}$ we define

$$i_\tau := \begin{array}{l}\text{value of } i \geq 0 \text{ at } (*) \text{ or } (**), \text{ depending on which}\\ \text{of these two points is first reached after } t \text{ has risen}\\ \text{to the value } \tau\end{array} \quad . \tag{11}$$

For every 2-set S we define

$$\mathbf{1}_S \colon \mathcal{G}_n \to \{0,1\}, \qquad G \longmapsto \mathbf{1}_S(G) := \begin{cases} 1 & \text{if } S \in \mathrm{E}(G) \\ 0 & \text{if } S \notin \mathrm{E}(G) \end{cases} \quad . \tag{12}$$

Then the following 'loop invariants' hold:

(li.0) $|\mathrm{U}_i^{(H)}| + |\mathrm{P}_i^{(H)}| + |\mathrm{F}_i^{(H)}| = n$ for every $i \geq 0$,
(li.1) $|\mathrm{F}_{i_\tau}^{(H)}| \cdot |\mathrm{U}_{i_\tau}^{(H)}| \leq \tau$ for every $1 \leq \tau \leq \binom{|H|}{2}$,
(li.2) $|\mathrm{F}_{i_\tau}^{(H)}| + |\mathrm{P}_{i_\tau}^{(H)}| > \sum_{1 \leq \tilde{t} \leq \tau} \mathbf{1}_{\mathrm{S}_{\tilde{t}}^{(H)}}(H)$ for every $1 \leq \tau \leq \binom{|H|}{2}$,
(li.3) $|\mathrm{P}_{i_\tau}^{(H)}| \leq 1 + \sum_{1 \leq \tilde{t} \leq \tau} \mathbf{1}_{\mathrm{S}_{\tilde{t}}^{(H)}}(H)$ for every $1 \leq \tilde{t} \leq \binom{|H|}{2}$.

Proof of Lemma 3.8. As to (li.0), since this property is preserved each time the sets U_i, P_i, F_i are modified, for each i we have the partition $\mathrm{U}_i^{(H)} \dot\cup \mathrm{P}_i^{(H)} \dot\cup \mathrm{F}_i^{(H)} = \mathrm{V}(H)$, which implies (li.0).

As to (li.1), for every $x \in \mathrm{F}_{i_t}^{(H)}$ we define

$$i_x := \min\{0 \leq \imath \leq i_t \colon x \in \mathrm{F}_\imath^{(H)}\} \quad . \tag{13}$$

Since $|\mathrm{U}_i^{(H)}|$ by construction of the algorithm is non-increasing w.r.t. increasing i, we have, for every $1 \leq \tau \leq \binom{|H|}{2}$ and every $x \in \mathrm{F}_{i_\tau}^{(H)}$,

$$|\mathrm{U}_{i_x}^{(H)}| \geq |\mathrm{U}_{i_t}^{(H)}| \quad . \tag{14}$$

For every $x \in \mathrm{F}_{i_\tau}^{(H)}$, for x to have been declared an element of $\mathrm{F}_{i_x}^{(H)}$ by the algorithm, all $|\mathrm{U}_{i_x}^{(H)}|$ elements $\tilde{v} \in \mathrm{U}_{i_x}^{(H)}$ must have been involved in a negative test for being an edge of H, and it follows from the construction of (A.1) that

$$\sum_{x \in \mathrm{F}_{i_\tau}^{(H)}} |\mathrm{U}_{i_x}^{(H)}| \leq \tau \quad . \tag{15}$$

Therefore

$$\tau \overset{15}{\geq} \sum_{x \in \mathrm{F}_{i_\tau}^{(H)}} |\mathrm{U}_{i_x}^{(H)}| \overset{(14)}{\geq} \sum_{x \in \mathrm{F}_{i_\tau}^{(H)}} |\mathrm{U}_{i_\tau}^{(H)}| = |\mathrm{U}_{i_\tau}^{(H)}| \cdot |\mathrm{U}_{i_\tau}^{(H)}| \quad , \tag{16}$$

which is (li.1).

The claims (li.2) and (li.3) can be proved by an inductive argument which distinguishes cases according to whether the last value of t was incremented in the inner if-clause or in the else-clause.

We are now sufficiently prepared to prove Theorem 3.7. For lack of time, in this lecture we leave the (simpler) proof of Theorem 3.7.(1) as an exercise for you (use a loop invariant and Lemma 3.5.1) and immediately go for the second part, i.e. Theorem 3.7.(2):

Summary of the proof of Theorem 3.7.2:

(St.1) Define of a suitable set $\mathcal{G}_{n,\mathrm{typ}} \subseteq \mathcal{G}_n$ of 'typical' elements of $\mathcal{G}_n$,

(St.2) Prove that

$$\mathsf{Pr}_{\mathrm{G}(n,\frac{1+\varepsilon}{n})}[\mathcal{G}_{n,\mathrm{typ}}] \xrightarrow{n\to\infty} 1 \tag{17}$$

(St.3) Prove the deterministic statement that there exists n_ε such that for every $n \geq n_\varepsilon$,

$$\mathcal{G}_{n,\mathrm{typ}} \subseteq \{G \in \mathcal{G}_n \colon G \text{ contains a path of length} \geq \tfrac{1}{5}\varepsilon^2 n\} \quad . \tag{18}$$

Steps (St.1)—(St.3) combined obviously imply Theorem 3.7.2.

For step (St.1) we define

$$\mathcal{G}_{n,\mathrm{typ}} := \left\{ G \in \mathcal{G}_n \colon \sum_{1\leq i\leq \frac{1}{2}\varepsilon n^2} \mathbf{1}_{\mathrm{S}_i^{(G)}}(G) \leq \tfrac{1}{2}\varepsilon(1+\varepsilon)n + n^{\frac{1+\varepsilon}{2}} \right\} \quad . \tag{19}$$

For step (St.2) we make use of what one might view as the key qualitative insight in the proof of Krivelevich and Sudakov: although the sequence $\mathrm{S}_1^{(G)},\ldots,\mathrm{S}_{\frac{1}{2}\varepsilon n^2}^{(G)}$ computed by the algorithm depends, as a sequence of 2-sets, in a complicated way on G, the *distribution* of the function

$$\sum_{1\leq i\leq \frac{1}{2}\varepsilon n^2} \mathbf{1}_{\mathrm{S}_i^{(\cdot)}}(\cdot) \colon \mathcal{G}_n \to \mathbb{Z}_{\geq 0}$$

is just the binomial distribution: directly from the definition of $\mathrm{G}(n,p_n)$ it follows that with $X_1,\ldots,X_{\frac{1}{2}\varepsilon n^2}$ denoting independent random variables with $\mathsf{Pr}[X_i = 1] = p_n$ and $\mathsf{Pr}[X_i = 0] = 1-p_n$, we have, for every $0 \leq y \leq \frac{1}{2}\varepsilon n^2$,

$$\mathsf{Pr}_{\mathrm{G}(n,p_n)}\left[\left\{G \in \mathcal{G}_n \colon \sum_{1\leq i\leq \frac{1}{2}\varepsilon n^2} \mathbf{1}_{\mathrm{S}_i^{(G)}}(G) = y\right\}\right] = \mathsf{Pr}\left[|\{i \in [\tfrac{1}{2}\varepsilon n] \colon X_i = 1\}| = y\right] \tag{20}$$

Therefore, and because of $\frac{1}{2}\varepsilon(1+\varepsilon)n = \frac{1}{2}\varepsilon n^2 \frac{1+\varepsilon}{n} = \mathsf{E}[X_i]$, the statement (17) follows from (20) and inequality (3) in Lemma 3.5, completing step (St.2).

For step (St.3), let an arbitrary $0 < \varepsilon < \varepsilon_0$ be given. Let n_ε be large enough so that for every $n \geq n_\varepsilon$,

(Prop.n_ε.1) $1 + n^{\frac{1+\varepsilon}{2}} + \frac{1}{2}\varepsilon(1+\varepsilon)n < \varepsilon n$,

(Prop.n_ε.2) $\frac{1}{30}\varepsilon^2 n^2 - \frac{1}{2}n^{\frac{3}{2}+\frac{\varepsilon}{2}} > 0$.

Such an n_ε obviously exists.

We now complete step (St.3) by showing that with this n_ε the inclusion (18) holds.

Let $n \geq n_\varepsilon$ and let $G \in \mathcal{G}_{n,\mathrm{typ}}$ be arbitrary.

$$\mathcal{G}_{n,\mathrm{typ}} \subseteq \{G \in \mathcal{G}_n \colon \text{in } G \text{ there is a path of length} \geq \tfrac{1}{5}\varepsilon^2 n\} \quad . \tag{21}$$

Let $n \geq n_\varepsilon$ and $G \in \mathcal{G}_{n,\mathrm{typ}}$ be arbitrary. We use the abbreviation

$$t_0 := \tfrac{1}{2}\varepsilon n^2 \quad . \tag{22}$$

Claim 1. $|\mathrm{F}_{i_{t_0}}^G| < \frac{1}{2}\varepsilon n + \varepsilon^2 n$.

Proof of Claim 1. For the sake of contradiction assume that $|\mathrm{F}_{i_{t_0}}^G| \geq \frac{1}{2}\varepsilon n + \varepsilon^2 n$. Then there exists

$$t_{\lightning} := \max\{1 \leq t \leq t_0 \colon |\mathrm{F}_{i_t}^{(G)}| = \tfrac{1}{2}\varepsilon n + \varepsilon^2 n\} \quad , \tag{23}$$

and we have

$$
\begin{aligned}
& |\mathrm{U}^{(G)}_{i_{t_\not{z}}}| \overset{(\text{li}.0)}{=} n - |\mathrm{P}^{(G)}_{i_{t_\not{z}}}| - |\mathrm{F}^{(G)}_{i_{t_\not{z}}}| \\
\text{(by definition of } t_\not{z}) \quad & = (1 - \tfrac{\varepsilon}{2} - \varepsilon^2)n - |\mathrm{P}^{(G)}_{i_{t_\not{z}}}| \\
\text{(by (li.3) in Lemma 3.8 with } H := G \text{ and } t := t_\not{z}) \quad & \geq (1 - \tfrac{\varepsilon}{2} - \varepsilon^2)n - (1 + \sum_{1 \leq t \leq t_\not{z}} \mathbf{1}_{\mathrm{S}^{(G)}_t}(G)) \\
\text{(since } t_0 \geq t_\not{z}) \quad & \geq (1 - \tfrac{\varepsilon}{2} - \varepsilon^2)n - (1 + \sum_{1 \leq t \leq t_0} \mathbf{1}_{\mathrm{S}^{(G)}_t}(G)) \\
\text{(since } G \in \mathcal{G}_{n,\text{typ}}) \quad & \geq (1 - \tfrac{\varepsilon}{2} - \varepsilon^2)n - (1 + n^{\frac{1+\varepsilon}{2}} + \tfrac{1}{2}\varepsilon(1+\varepsilon)n) \\
\text{(by (Prop.} n_\varepsilon.1)) \quad & > (1 - \tfrac{\varepsilon}{2} - \varepsilon^2)n - \varepsilon n \\
& = (1 - \tfrac{3}{2}\varepsilon - \varepsilon^2)n \quad , \qquad (24)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
& t_\not{z} \overset{\text{Lemma 3.7.1}}{\geq} |\mathrm{F}_{i_{t_\not{z}}}| \cdot |\mathrm{U}_{i_{t_\not{z}}}| \\
\text{(definition of } t_\not{z}) \quad & = (\tfrac{1}{2}\varepsilon n + \varepsilon^2 n) \cdot |\mathrm{U}_{t_\not{z}}| \\
\text{(by (24))} \quad & > (\tfrac{1}{2}\varepsilon + \varepsilon^2)(1 - \tfrac{3}{2}\varepsilon - \varepsilon^2)n^2 \\
& = (\tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon^2 - 2\varepsilon^3 - \varepsilon^4)n^2 \\
\text{(by (Prop.} \varepsilon_0.(2))) \quad & > \tfrac{1}{2}\varepsilon n^2 = t_0 \geq t_\not{z} \quad , \qquad (25)
\end{aligned}
$$

contradiction. This shows that $|\mathrm{F}^{G}_{i_{t_0}}| \geq \frac{1}{2}\varepsilon n + \varepsilon^2 n$ cannot be true.

Claim 2. $|\mathrm{P}^{(G)}_{i_{t_0}}| \geq \frac{1}{5}\varepsilon^2 n$. For the sake of contradiction assume that

$$
|\mathrm{P}^{(G)}_{i_{t_0}}| < \tfrac{1}{5}\varepsilon^2 n \quad . \qquad (26)
$$

Then with t_0 as in (22)

$$
\begin{aligned}
& |\mathrm{F}_{i_{t_0}}| \overset{(\text{li}.2)}{>} \sum_{1 \leq \tilde{t} \leq t} \mathbf{1}_{\mathrm{S}^{(H)}_{\tilde{t}}}(H) - |\mathrm{P}^{(H)}_i| \\
\text{(since } G \in \mathcal{G}_{n,\text{typ}}) \quad & \geq \tfrac{1}{2}\varepsilon(1+\varepsilon)n - n^{\frac{1+\varepsilon}{2}} - |\mathrm{P}^{(H)}_i| \\
\text{(by (26))} \quad & > \tfrac{1}{2}\varepsilon(1+\varepsilon)n - n^{\frac{1+\varepsilon}{2}} - \tfrac{1}{5}\varepsilon^2 n \\
& = \tfrac{1}{2}\varepsilon n + \tfrac{3}{10}\varepsilon^2 n - n^{\frac{1+\varepsilon}{2}} \quad , \qquad (27)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
& \tfrac{1}{2}\varepsilon n^2 = t_0 \overset{((\text{li}.0) \text{ and } (\text{li}.1) \text{ in Lemma 3.7})}{\geq} |\mathrm{F}^{(G)}_{i_{t_0}}| \cdot (n - |\mathrm{P}^{(G)}_{i_{t_0}}| - |\mathrm{F}^{(G)}_{i_{t_0}}|) \\
\text{(by (26), (27) and Claim 1)} \quad & > (\tfrac{1}{2}\varepsilon n + \tfrac{3}{10}\varepsilon^2 n - n^{\frac{1+\varepsilon}{2}})(n - \tfrac{1}{5}\varepsilon^2 n - \tfrac{1}{2}\varepsilon n - \varepsilon^2 n) \\
& = \tfrac{1}{2}\varepsilon n^2 + (\tfrac{1}{20}\varepsilon^2 - \tfrac{3}{4}\varepsilon^2 - \tfrac{9}{25}\varepsilon^4)n^2 - (1 - \tfrac{1}{2}\varepsilon - \tfrac{6}{5}\varepsilon^2)n^{\frac{3}{2}+\frac{\varepsilon}{2}} \\
\text{(because of (Prop.} \varepsilon_0.(1) \text{ and (Prop.} \varepsilon_0.(3))) \quad & > \tfrac{1}{2}\varepsilon n^2 + \tfrac{1}{30}\varepsilon^2 n^2 - \tfrac{1}{2}n^{3/2+\varepsilon/2} \\
\text{(because of (Prop.} n_\varepsilon.2)) \quad & > \tfrac{1}{2}\varepsilon n^2 \quad , \qquad (28)
\end{aligned}
$$

a contradiction. This shows that (26) cannot be true.

To complete step (St.3) it now suffices to note that by construction of the depth-first-search computing $\mathrm{P}_i(G)$, the graph $G[\mathrm{P}_i(G)]$ induced by this set contains a Hamilton path, for every i, and in particular for $i = i_{t_0}$, so Claim 2 implies (18), completing the proof.