Lecture 11, 4.7.12

Topic: - inside the phase transition

- last week: component size at $p = \frac{1}{n}$
- today: subgraphs

First Goal:

<u>Thm 3.9</u> Let $t = t(n) := 1/n$. Then

$$\Pr[K_3 \subset G(n,p)] \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } p/t \to 0 \\ 1 - e^{-\frac{c^3}{6}} & \text{if } p/t \to c \\ 1 & \text{if } p/t \to \infty \end{cases}$$

0- and 1- statement already in Thm 1.4 or 3.4.

Need a new tool from „probability theory"

Thm 3.10 (Janson's inequality)

As in Lemma 1.12: $Y = \sum_{i \in I} X_i$, $X_i \in \{0,1\}$, $\mu = Ex[Y]$

write $i \sim j$ iff events $X_i = 1$ and $X_j = 1$ not independant

$$\Delta := \sum_{(i,j) : i \neq j} Pr[(X_i = 1) \wedge (X_j = 1)]$$

$$\tilde{\Delta} := \sum_{(i,j) : i \sim j} Pr[(X_i = 1) \wedge (X_j = 1)]$$

Recall from 1.12: $Pr[Y = 0] \leq \frac{1}{\mu} + \frac{\Delta}{\mu^2} - 1$ (1)

$Pr[Y = 0] \leq \frac{1}{\mu} + \frac{\tilde{\Delta}}{\mu^2}$ (2)

Let's assume in addition that $\forall i \in I$ $Pr[X_i = 1] \longrightarrow 0$,)

$$\forall J \subset I,\ \forall i \notin J: \quad \Pr\left[X_i = 1 \;\middle|\; \bigwedge_{j \in J} (X_j = 0)\right] \leq \Pr[X_i = 1] \tag{3}$$

$$\forall J \subset I,\ \forall i,k \notin J: \quad \Pr\left[(X_i = 1) \wedge (X_k = 1) \;\middle|\; \bigwedge_{j \in J} (X_j = 0)\right] \leq \Pr\left[(X_i = 1) \wedge (X_k = 1)\right].$$

Then:

$$e^{-\mu} \leq \Pr[Y = 0] \leq \begin{cases} e^{-\mu + \tilde{\Delta}} & \text{always} \\ e^{-\frac{\mu^2}{2\tilde{\Delta}}} & \text{if } \tilde{\Delta} \geq \mu. \end{cases} \tag{4}$$

Remark 3.11

a) Set $p_i := \Pr[X_i = 1] \longrightarrow 0$.

Then $\mu = \mathrm{Ex}[Y] = \sum \mathrm{Ex}[X_i] = \sum p_i$.

If all X_i were pairwise independent, then

$$\Pr[Y = 0] = \prod_{i \in I} \Pr[X_i = 0] = \prod_i (1 - p_i) \sim \prod e^{-p_i} = e^{-\sum p_i} = e^{-\mu}$$

b) Compare $\Pr[Y=0] \le \begin{cases} \frac{1}{\mu} + \frac{\tilde{\Delta}}{\mu^2} & \text{by (2)} \\ e^{-\frac{\mu^2}{\tilde{\Delta}}} & \text{by (4)} \end{cases}$.

If $\frac{\mu^2}{\tilde{\Delta}} \longrightarrow \infty$, then (3) is exponentially small compared to (2).

<u>Proof of Thm 3.9</u>

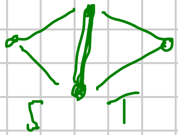
$p/t = pn \longrightarrow c$.

For $S \in \binom{[n]}{3}$ define $X_S = \begin{cases} 1 & \text{if } A_S := \left(G(n,p)[S] \cong K_3 \right) \\ 0 & \text{if } \overline{A_S} \end{cases}$

$\Rightarrow \quad Y := \sum_S X_S = \#$ triangles in $G(n,p)$, $\quad \Pr[A_S] = p^3$

$\Rightarrow \quad \mu := \mathrm{Ex}[X_S] = \binom{n}{3} p^3 \sim \frac{n^3}{6} p^3 \longrightarrow \frac{c^3}{6}$

and $\tilde{\Delta} := \sum_{S\sim T} \Pr[A_S \sim A_T] \le n^4 p^5 = (\underbrace{np}_{\to c})^4 \underbrace{p}_{\to 0} \longrightarrow 0$



Check conditions in (3) in Thm 3.10 hold:

absence of triangles in some place doesn't increase probability for triangles elsewhere.

$$(4) \Rightarrow \quad e^{-\mu} \le \Pr[Y=0] \le e^{-\mu+\tilde{\Delta}}$$

$$\Rightarrow \quad 1 \le \frac{\Pr[Y=0]}{e^{-\mu}} \le e^{\tilde{\Delta}} \xrightarrow[\text{because } \tilde{\Delta}\to 0]{n\to\infty} 1$$

$$\Rightarrow \quad \Pr[Y=0] \xrightarrow{n\to\infty} e^{-\mu} \to e^{-\frac{c^3}{6}}$$

$\square$

What if we have a sharp threshold?

<u>Thm 3.12</u> Let $t = \frac{\ln(n)}{n}$. Then $\forall \varepsilon > 0$

$$Pr\left[\, G(n,p) \text{ has no isolated vertex} \,\right] \xrightarrow{n\to\infty} \begin{cases} 0 & p/t \leq 1-\varepsilon \\ e^{-e^{-c}} & p/t \to 1 + \frac{c}{\ln(n)} \\ 1 & p/t \geq 1+\varepsilon \end{cases}$$

(or: is connected)

<u>Proof</u>: homework!

4. Clique number & Chromatic number of random graphs

Prop 4.1

a) $$m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}$$ (Stirling's formula)

b) $$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

c) $$\log \binom{n}{k} = k \log n - k \log k + O(k)$$

Proof: (boring)

$\omega(G) :=$ size of a largest clique in G

Thm 4.2 $\exists$ function $\ell_0 = \ell_0(n) \sim 2 \log n$ (note: $\log = \log_2$), such that

a) $\Pr[\ell_0 - 1 \leq \omega(G(n, \tfrac{1}{2})) \leq \ell_0] \longrightarrow 1$

b) $\Pr[\omega(G(n, \tfrac{1}{2})) < \ell_0 - 4] \leq e^{-n^{2-o(1)}}$.

Proof: r.v. $Y_\ell :=$ # of ℓ-cliques in $G(n, \tfrac{1}{2})$

to prove a), it suffices to find a function $\ell_0 \sim 2 \log n$ such that

- $\Pr[Y_{\ell_0+1} \geq 1] \xrightarrow{n \to \infty} 0$ (5)
- $\Pr[Y_{\ell_0-1} = 0] \xrightarrow{n \to \infty} 0$ (6).

Observe $\mathrm{Ex}[Y_1] = n$, $\mathrm{Ex}[Y_n] = 2^{-\binom{n}{2}}$,

so choose the largest ℓ such that $\mathrm{Ex}[Y_\ell] \geq \frac{1}{\sqrt{n}}$

and call it ℓ_0.

$\underset{\text{Constr.}}{\Rightarrow}$ $\mathrm{Ex}[Y_{\ell_0+1}] < \frac{1}{\sqrt{n}}$

$\Rightarrow$ $\Pr[Y_{\ell_0+1} \geq 1] \underset{\text{Markov}}{\leq} \mathrm{Ex}[Y_{\ell_0+1}] < \frac{1}{\sqrt{n}} \longrightarrow 0$ $\Rightarrow (5)$

Let's check whether $\ell_0 \sim 2 \log n$. We have

$$\mathrm{Ex}[Y_\ell] = \binom{n}{\ell} 2^{-\binom{\ell}{2}} \overset{4.1\,c)}{=} 2^{\ell \log n - \ell \log \ell + O(\ell) - \frac{\ell^2}{2} + \frac{\ell}{2}}$$

$$= 2^{\ell \log n - \ell^2/2 + o(\ell^2)} \qquad (7)$$

Note: to decide whether $\mathrm{Ex}[Y_\ell] \to 0$ or $\to +\infty$,

we need to decide whether $\ell \log n \overset{>}{\underset{=}{<}} \ell^2/2$

$\ell \log n = \ell^2/2 \iff 2 \log n = \ell$

Set $\ell \sim c \log n$ then

$$\mathrm{Ex}[Y_\ell] \overset{(7)}{=} 2^{c \log^2 n - \frac{c^2}{2} \log^2 n + o(\log^2 n)} = n^{c \log n \,(1 - \frac{c}{2}) + o(\log n)}$$

For $c > 2+\varepsilon$: $\mathrm{Ex}[Y_\ell] < \frac{1}{n}$,

For $c < 2-\varepsilon$: $\mathrm{Ex}[Y_\ell] > n$.

Since $\mathrm{Ex}[Y_{\ell_0+1}] < \frac{1}{\sqrt{n}} \leq \mathrm{Ex}[Y_{\ell_0}]$,

we must have $\ell_0 \sim 2 \log(n)$.

Next step: Analyse the ratio $\mathrm{Ex}[Y_{\ell+1}] / \mathrm{Ex}[Y_\ell]$.

$$\frac{\mathrm{Ex}[Y_{\ell+1}]}{\mathrm{Ex}[Y_\ell]} = \frac{\binom{n}{\ell+1} 2^{-\binom{\ell+1}{2}}}{\binom{n}{\ell} 2^{-\binom{\ell}{2}}} = \frac{n-\ell}{\ell+1} 2^{-\ell} < n 2^{-\ell},$$

So when $\ell \sim 2 \log n$ we have

$$\frac{\mathrm{Ex}[Y_{\ell+1}]}{\mathrm{Ex}[Y_\ell]} < n 2^{-\ell} = n 2^{-(2+o(1)) \log n} = n \, n^{-2-o(1)} = \frac{1}{n^{1+o(1)}}. \quad (8)$$

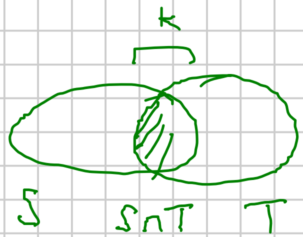
So expectation drops (at least) by a factor n.

So let's now prove (6) : $\Pr[Y_{\ell_0 - 1} = 0] \longrightarrow 0$. Try 2nd moment.

Let $\ell \sim 2 \log n$, $\quad \mu = \mathrm{Ex}[Y_\ell]$, $\quad S \in \binom{[n]}{\ell}$

$$X_S := \begin{cases} 1 & G(n,p)[S] \cong K_\ell \\ 0 & \text{otherws} \end{cases}, \qquad Y_\ell = \sum_{S \in \binom{[n]}{\ell}} X_S$$

$$\tilde{\Delta} = \sum_{S \sim T} \Pr\left[(X_S = 1) \wedge (X_T = 1) \right]$$



$$= \sum_{k=2}^{\ell-1} \sum_{|S \cap T| = k} 2^{-\left(2\binom{\ell}{2} - \binom{k}{2}\right)}$$

$$= \sum_{k=2}^{\ell-1} \binom{n}{\ell} \cdot \binom{\ell}{k} \binom{n-\ell}{\ell-k} 2^{-2\binom{\ell}{2} + \binom{k}{2}}$$

$$\Rightarrow \frac{\tilde{\Delta}}{\mu^2} = \sum_{k=2}^{\ell-1} \frac{\binom{n}{\ell}\binom{\ell}{k}\binom{n-\ell}{\ell-k} 2^{-2\binom{\ell}{2} + \binom{k}{2}}}{\binom{n}{\ell}^2 2^{-2\binom{\ell}{2}}} = \sum_{k=2}^{\ell-1} \frac{\binom{\ell}{k}\binom{n-\ell}{\ell-k} 2^{\binom{k}{2}}}{\binom{n}{\ell}} =: \sum_{k=2}^{\ell-1} g(k)$$

A rather boring computation shows that $\sum_{k=2}^{\ell-1} g(k) \sim g(2)$. Hence

$$\Rightarrow \frac{\tilde{\Delta}}{\mu^2} \sim g(2) = \frac{\frac{\ell(\ell-1)}{2}(n-\ell)^{\ell-2} \cdot 2 \cdot \ell!}{n^{\ell}\,(\ell-2)!} \leq \frac{\ell^4}{(n-\ell)^2} \longrightarrow 0 \qquad (9)$$

Now for $\ell = \ell_0 - 1$

$$\mu = \mathrm{Ex}[Y_{\ell_0-1}] \overset{(8)}{>} n^{1+o(1)}\,\mathrm{Ex}[Y_{\ell_0}] \geq \frac{n^{1+o(1)}}{\sqrt{n}} = n^{\frac{1}{2}+o(1)} \longrightarrow \infty$$

So we get $\Pr[Y_{\ell_0-1} = 0] \underset{(2)}{\leq} \frac{1}{\mu} + \frac{\tilde{\Delta}}{\mu^2} \longrightarrow 0$

$\mu \to \infty$, $\frac{\tilde{\Delta}}{\mu^2} \to 0$

qed for a).