Notiztitel 10.07.2012

Lecture 12, 11.7.12

Thm 4.2 $\exists$ function $l_0 = l_0(n) \sim 2 \log n$ such that

a) $\Pr[\, l_0 - 1 \le \omega(G(n, \tfrac{1}{2})) \le l_0 \,] \longrightarrow 1$.

b) $\Pr[\, \omega(G(n, \tfrac{1}{2})) < l_0 - 4 \,] \le e^{-n^{2-o(1)}}$.

Proof: a) done:

Recall: Key idea: $\mathrm{Ex}[Y_{l_0+1}] \le \frac{1}{\sqrt{n}} \le \mathrm{Ex}[Y_{l_0}]$

for $l \sim 2 \log n$: $\quad \frac{\mathrm{Ex}[Y_{l+1}]}{\mathrm{Ex}[Y_l]} < \frac{1}{n^{1+o(1)}}$ (8)

2nd moment: $Y_l := \sum_{S \in \binom{[n]}{l}} X_S$, $\quad \mu := \mathrm{Ex}[Y_l]$

$\tilde{\Delta} := \sum_{S \sim T} \Pr[\,(X_S = 1) \wedge (X_T = 1)\,]$

Then $\mu = \binom{n}{\ell} 2^{-\binom{\ell}{2}}$ and $\frac{\tilde{\Delta}}{\mu^2} \sim \frac{\ell^4}{(n-\ell)^2}$ (9)

Now we prove b): $\ell := \ell_0 - 4$ Aim: $\Pr[Y_{\ell_0 - 4} = 0] \leq e^{-n^{2-o(1)}}$

Plan: $\Pr[Y = 0] \underset{3.10(4)}{\leq} e^{-\mu^2/(2\tilde{\Delta})}$ if $\tilde{\Delta} \geq \mu$

$$\mu = \mathbb{E}[Y_{\ell_0 - 4}] \overset{8}{>} n^{4+o(1)} \cdot \mathbb{E}[Y_{\ell_0}] \geq n^{3.5+o(1)} \longrightarrow \infty$$

and $\frac{\tilde{\Delta}}{\mu} = \frac{\tilde{\Delta}}{\mu^2} \cdot \mu \overset{(9)}{\gtrsim} \frac{\ell^4}{(n-\ell)^2} n^{3.5+o(1)} \geq 1 \Rightarrow \tilde{\Delta} \geq \mu$

Moreover, $\frac{\mu^2}{2\tilde{\Delta}} \overset{(9)}{\sim} \frac{(n-\ell)^2}{2\ell^4} \geq n^{2-o(1)}$ because $\frac{n^2}{\log n} \underset{\forall \varepsilon}{\geq} \frac{n^2}{n^\varepsilon} = n^{2-\varepsilon}$

So the plan works. $\square$

Now: $\chi(G(n,\frac{1}{2}))$.

History:

1960s: $\chi(G(n,\frac{1}{2})) \geq \dfrac{n}{\alpha(G(n,\frac{1}{2}))} = \dfrac{n}{\omega(G(n,\frac{1}{2}))} \overset{4.2}{\sim} \dfrac{n}{2\log n}$.

1970s: $\chi_{Greedy}(G(n,\frac{1}{2})) \overset{4.5}{\sim} \dfrac{n}{\log(n)}$.

idea: pull out inclusion-maximal independent sets

1980s: $\chi(G(n,\frac{1}{2})) \overset{4.3}{\leq} (1+\varepsilon)\dfrac{n}{2\log(n)}$

all statement a.a.s $\hat{=}$ with $\Pr[\cdot] \xrightarrow{n\to\infty} 1$.

open: find an algorithm which uses $< \dfrac{n}{\log(n)}$ colours.

Thm 4.3 $$\Pr\left[\chi(G(n,\tfrac{1}{2})) \le (1+o(1))\frac{n}{2\log n}\right] \xrightarrow{n\to\infty} 1$$

Proof:

Denote by $\mathcal{A}$ the following property of a graph G:

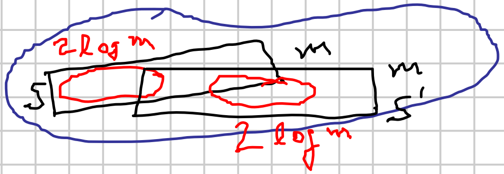
Every set $S \subset [n]$ of size $m := \frac{n}{\log^2 n}$ contains an independent set of size $r := k_0(m) - 4$

(i.e. $r \sim 2\log m$)

Claim 1: $\Pr[G(n,\frac{1}{2}) \text{ has } \mathcal{A}] \xrightarrow{n\to\infty} 1$.

Claim 2: If a graph G has $\mathcal{A}$, then $\chi(G) \le (1+o(1))\frac{n}{2\log(n)}$.

Proof of Claim 1: $G(n,\frac{1}{2})[S] = G(m,\frac{1}{2})$



! to have an indep set of size r is not indep for two sets S and S'

but: $\Pr[\alpha(G(n,\tfrac{1}{2})[S]) < r] = \Pr[\omega(\overline{G(n,\tfrac{1}{2})[S]}) < r] < e^{-m^{2-o(1)}}$

$\#$ sets $S = \binom{n}{m} \le 2^n$

$\Rightarrow \Pr[G(n,\tfrac{1}{2}) \text{ does not have } \mathcal{A}] = \Pr[\exists S \in \binom{[n]}{m} : \alpha(G(n,\tfrac{1}{2})[S]) < r]$

$\le 2^n e^{-m^{2-o(1)}} \le e^{n - \left(\frac{n}{\log^2 n}\right)^{2-o(1)}} \xrightarrow[m\to\infty]{} 0$

Proof of Claim 2:

Suppose G has property $\mathcal{A}$. Construct a colouring of G as follows:

a) choose an arbitrary $S \in \binom{[n]}{m}$.

b) take independent set of size r in S and give all its vertices a new colour, then delete coloured vertices

Known to exist because of $\mathcal{A}$

c) if at least m vertices remain, go to a)
otherwise give all remaining vertices a new, individual colour and stop.

$$\Rightarrow \# \text{ of colours used } \leq \frac{n}{T} + m$$

$$\sim \frac{n}{2\log m} + \frac{n}{\log^2 n}$$

$$\leq \frac{n}{2\log\left(\frac{n}{\log^2 n}\right)} + \frac{n}{\log^2 n}$$

$$= \frac{n}{2\log n - 2\log(\log^2 n)} + \frac{n}{\log^2 n}$$

$$\sim \frac{n}{2\log n}.$$

qed

Why is this not an „algorithm"?