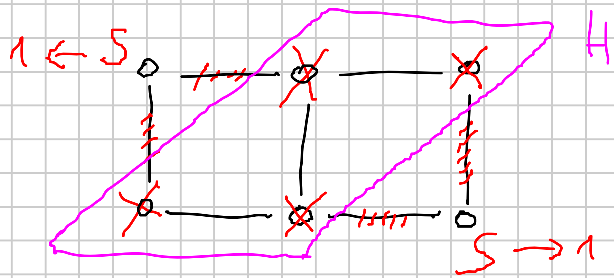
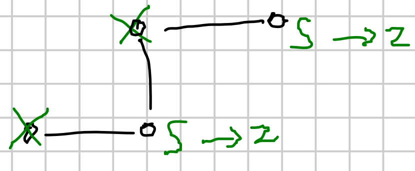
recall: 4.2/4.3: $\chi(G(n,\frac{1}{2})) \sim \frac{n}{2\log n}$ Lecture 13, 18.7.12

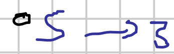
Algo 4.4 Greedy-Colouring (G)



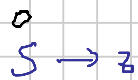
1) $k := 0$
2) repeat
3) $\quad S := \emptyset$
4) $\quad H := G$
5) $\quad$ while $H \neq \emptyset$ do



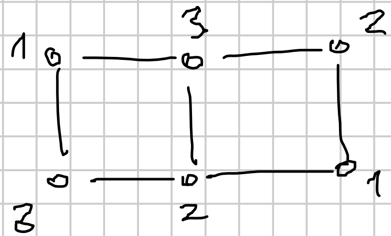
6) $\quad\quad S := S + x$ for an arbitrary vertex x in H
7) $\quad\quad H := H - x - N_H(x)$



8) $\quad k := k+1$



9) $\quad$ colour all vertices in S with colour k
10) $\quad G := G - S$
11) until $G = \emptyset$



Thm 4.5 Greedy-Colouring always produces a legal colouring and $\Pr\left[\text{it needs} \le (1+o(1))\frac{n}{\log n} \text{ colours}\right] \xrightarrow{n\to\infty} 1$.

essential fact (for the proof): the graph H defined in the beginning of each repeat-loop (line 4) is still a random graph.

" method of deferred decision"

Claim: Assume that we are at the beginning of a repeat-loop and that the remaining graph $H=(V,E)$ has $m=|V|$ vertices. Let S be the indep. set generated in this loop and $s:=|S|$. Then $\forall \varepsilon>0$

$$\Pr\left[s \le (1-\varepsilon)\log m\right] \le e^{(\ln m)(\log m) - (1-o(1))m^{\varepsilon}}.$$

Proof of Claim:

Note that S is maximal independent:  $\forall x \in V\setminus S \ \exists y \in N(x)\cap S$

Fix an arbitrary set $\hat{S} \subset V$ with $|\hat{S}| = s \le (1-\varepsilon)\log m$.

Let $\hat{T} := V \setminus \hat{S}$ and $t := |\hat{T}|$.

$$\Pr[\,x \text{ has no neighbour in } S\,] = 2^{-s}$$

$$\Pr[\,x \text{ has at least one neighbour in } S\,] = 1 - 2^{-s}$$

$$\Pr[\,\text{all } x \in V\setminus\hat{S} \text{ have at least one neighb in } \hat{S}\,] = (1-2^{-s})^{t} \le e^{-2^{-s}t}$$

$$\Pr[\,\exists \text{ set } S \text{ such that: all } x \in V\setminus S \ldots\,] \le m^{s} e^{-2^{-s}t} = e^{(\ln m)s - 2^{-s}t}$$

Use assumption $s \le (1-\varepsilon)\log m$

$\Rightarrow t = m - s = m - o(m) = m(1-o(1))$

$$\Rightarrow \quad \le e^{(\ln m)(\log m) - 2^{-(1-\varepsilon)\log m} m(1-o(1))}$$

rough estimate

precise estimate

$$= e^{(\ln m)(\log m) - m^{-(1-\varepsilon)} m(1-o(1))} = e^{(\ln m)(\log m) - (1-o(1))m^{\varepsilon}}$$

qed (Claim).

Proof of Thm 4.5   While the graph has $m \geq \frac{n}{\log^2 n}$ verts left:

$$\Pr[\exists \text{ loop with a colour class of size } < (1-\varepsilon)\log m] \leq \Pr[\exists \text{ iteration } \exists \text{ maximal indep set} \ldots]$$

$$\leq n \cdot e^{(\ln m)(\log m) - (1-o(1))\, m^{\varepsilon}} = e^{\underbrace{\ln n + (\ln m)(\log m) - (1-o(1))\, m^{\varepsilon}}_{\to -\infty}}$$

because $\log m \geq \log\left(\frac{n}{\log^2 n}\right) = \log n - 2\log\log n = (1-o(1))\log n$. (*)

So, almost surely, we will find $(1-o(1))\log m \overset{(*)}{=} (1-o(1))\log n$ vertices in each colour class. In total:

$$\#\text{ colours used} \leq \frac{n}{(1-o(1))\log n} + \frac{n}{\log^2 n} = (1+o(1))\frac{n}{\log n}$$

Def 4.6 Let A be a colouring algorithm.

approx. ratio of A for input G $:= \dfrac{\#\text{ colours used by } A \text{ to colour } G}{\chi(G)} \geq 1$

Thm 4.7

If there exists a poly time colouring algorithm which for all inputs has approximation ratio $\leq n^{1-\varepsilon}$ then $P = NP$.

not with a guaranteed poly time!

Cor 4.8 to Thm 4.5

For almost all graphs Greedy-Colouring (Algo 4.4.) has approximation ratio ≤ 2.

⇝ open: improve factor 2?

⇝ hard instances seem to be „rare"

⇝ Can we turn this into an algo with guaranteed approx. ratio

idea:
- run greedy-colouring.
- if something goes wrong (happens with low prob.), run an expensive exact colouring algo
- still a good average running time and a guaranteed performance ...

What could go wrong?

a) algo could need more than $(1+\varepsilon)\, n / \log n$ colours

→ check this by counting them.

b) input graph may need much less than $\frac{n}{2 \log(n)}$ colours.

→ how can we tell ???

⇝ need lower bounds that can be computed fast.

Thm 4.9

There exists a colouring algorithm with expected poly running time and guaranteed approx. ratio $O(\sqrt{n})$ for inputs $G(n, \frac{1}{2})$.

Open problem 4.10 improve this to 1.