Abel's Theorem

During our studies of Analysis 1 in the first semester, we have discussed power series. A power series is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c)^1 + a_2 (x-c)^2 + \ldots$$

where $a_n$ represents the coefficient of the nth term and c is a constant.

Examples:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

These power series are also examples of Taylor series.

A power series will converge for some values of the variable $x$ and may diverge for others. All power series converge at $x = c$. If $c$ is not the only convergent point, then there is always a number $r$ with $0 < r \leq \infty$ such that the series converges whenever $\mid x - c \mid < r$ and diverges whenever $\mid x - c \mid > r$ . The number $r$ is called the radius of convergence of the power series.
In general, it is given as

$$r = \frac{1}{limsup_{n \to \infty} \mid a_n \mid^{\frac{1}{n}}}$$

This is the Cauchy-Hadamard theorem, which has been introduced during the first semester.

The series converges absolutely for $\mid x - c \mid < r$ and converges uniformly on every compact subset of $\{x : \mid x - c \mid < r\}$ .

For $\mid x - c \mid = r$, we cannot make any general statement on whether the series converges or diverges. However, for the case of real variables, Abel's theorem states that the sum of the series is continuous at $x$ if the series converges at $x$.

Abel's Theorem:

Let $\sum_{n=0}^{\infty} c_n$ be a convergent series of real numbers. Then, the power series $f(x) := \sum_{n=0}^{\infty} c_n$ converges uniformly on the interval $[0, 1]$. It therefore represents a continuous function on $[0, 1]$.

Remark: The German name for this Theorem is Abelscher Grenzwertsatz. Its connection to the mathematical limit can be seen from an alternative proof we provide in the appendix.

Proof: Analysis 1, Otto Forster 9. Auflage: Paragraph 21 Satz 3 und Paragraph 22 Satz 5.

Alternative Version and Proof of Abel's Theorem:

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $r > 0$ , $r \in \mathbb{R}$ . If the series converges for $x = r$, then the function defined on the interval $-1 < x \leq r$ is left-continuous in $r$, i.e.

$$lim_{x \to r-0} f(x) = lim_{x \to r-0} \sum_{n=0}^{\infty} a_n x^n = f(r) = \sum_{n=0}^{\infty} a_n r^n$$

Proof

W.l.o.g. (Short form for "Without loss of generality" :) ) let $r = 1$. This assumption is valid, because if $\sum_{n=0}^{\infty} a_n x^n$ has the radius of convergence $r$, then the series $\sum_{n=0}^{\infty} a'_n x^n$, where $a'_n = a_n r^n$, has a radius of convergence $p = 1$ and $\sum a'_n$ converges if and only if $\sum_{n=0}^{\infty} a_n r^n$ converges.

Thus $r = 1$ and $S := \sum_{n=0}^{\infty} a_n$ converges. We now proof

$$lim_{x \to 1-0} f(x) = \sum_{n=0}^{\infty} a_n = S$$

Preliminaries

The product of two convergent series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $r_f = 1$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with $r_g = 1$ is defined via the Cauchy-product for series:

$$f(x)g(x) = \sum_{n=0}^{\infty}(a_0 b_n + a_1 b_{n-1} + ... + a_n b_0)x^n$$

By setting $g(x) := \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (geometric series, $\mid x \mid < 1$), it follows that

$$\frac{1}{1-x} \cdot f(x) = \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_j x^n = \sum_{n=0}^{\infty} s_n x^n$$

, which is equivalent to

$$f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n \quad , \quad s_n = \sum_{j=0}^{n} a_j \qquad (*)$$

Keeping in mind $(*)$ it follows for $\mid x \mid < 1$

$$S - f(x) = S - (1-x) \sum_{n=0}^{\infty} s_n x^n$$

Now we need to express $S$ as an infite series. Using geometric series, we find

$$\frac{(1-x) \cdot S}{(1-x)} = (1-x) \cdot S \sum_{n=0}^{\infty} x^n = (1-x) \sum_{n=0}^{\infty} S x^n$$

Substituting this into the above equation we obtain

$$S - f(x) = (1-x) \sum_{n=0}^{\infty} (S - s_n) x^n = \sum_{n=0}^{\infty} r_n x^n, \quad r_n = \sum_{j=n+1}^{\infty} a_j \qquad (**)$$

Let $\epsilon > 0$ . It follows from the convergence of $\sum_{j=n+1}^{\infty} a_j$ :

$$\exists\, m \in \mathbb{N} : \mid r_n \mid = \mid \sum_{j=n+1}^{\infty} a_n \mid < \frac{\epsilon}{2} \quad \forall n > m$$

From (∗∗) it follows for $0 < x < 1$

$$| S - f(x) | < (1-x) \sum_{n=0}^{m} | r_n | x^n + (1-x) \frac{\epsilon}{2} \sum_{n=m+1}^{\infty} x^n < (1-x) \sum_{n=0}^{m} | r_n | + \frac{\epsilon}{2}$$

Since $\sum_{n=0}^{m} | r_n |$ has a fixed value $\forall \epsilon > 0 \exists\, \delta > 0$ such that

$$| 1 - x | < \delta \Rightarrow (1-x) \sum_{n=0}^{m} | r_n | < \frac{\epsilon}{2}$$

Summarizing the above, we showed

$$\forall \epsilon > 0 \;\; \exists \delta > 0 : | 1 - x | < \delta \Rightarrow | S - f(x) | < \epsilon$$

which proves the hypothesis.

IMC Aufgabe: IMC 2010 Day 1 Question 2 Solution 1 + Remark !