Cotangent and the Herglotz trick

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The following script introduces the partial fraction expression of the cotangent function and provides an elegant proof, using the Herglotz trick.

This expression of the cotangent is used in the second part of this script to solve an actual IMC problem. As the complex logarithm function which is needed in order to fully understand the IMC problem may not be familiar to everyone, its most important features will be summarized.

1 Partial Fraction Expansion of Cotangent Function

The partial fraction expression of the cotangent function was proven by Euler in his *Introductio in Analysin Infinitorum* from 1748 and is regarded as one of the most interesting formula involving elementary functions:

$$\begin{aligned} \pi \cot(\pi x) &= \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) \qquad (x \in \mathbb{R} \backslash \mathbb{Z}) \\ &= \lim_{N \to \infty} \sum_{n=-N}^{N} \left(\frac{1}{x+n} \right). \end{aligned}$$

We will prove this equation through the Herglotz trick. Let

$$f(x) := \pi \cot(\pi x) \qquad g(x) := \lim_{N \to \infty} \sum_{n=-N}^{N} \left(\frac{1}{x+n} \right).$$

First, we derive four common properties of those two functions. In a second step, we reason that $f(x)$ and $g(x)$ must be identical.

A) The functions $f(x)$ and $g(x)$ are defined for all non-integral values and are continuous there.

It is obvious that f is defined for all non-integral values, as $f(x) = \pi \cot(\pi x) = \pi \frac{\cos(\pi x)}{\sin(\pi x)}$. The function is continuous, as it is the quotient of two continuous functions, which denominator does not equal 0. The function $g(x)$ can be rearranged as

$$g(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{2x}{n^2 - x^2} \right).$$

We have to prove that the term $\sum_{n=1}^{\infty} \left(\frac{2x}{n^2-x^2} \right)$ converges uniformly in a neighborhood of x. The first term for $n = 1$ and the first $2n - 1 \leq x^2$ terms do not provide any problems since they are finite, whereas for $n \geq 2$ and $2n - 1 > x^2$, that is $n^2 - x^2 > (n-1)^2 > 0$, the summands are bounded by

$$0 < \frac{1}{n^2 - x^2} < \frac{1}{(n-1)^2}.$$

which is also true for values in the neighborhood of x. As the sum $\sum_{n=1}^{\infty} \frac{1}{(n-1)^2}$ converges to $\frac{\pi^2}{6}$, the function g is defined and continuous.

B) The functions $f(x)$ and $g(x)$ are periodic of period 1.

The function f is periodic of period 1, as the cotangent has period π. Let

$$g_N(x) = \sum_{n=-N}^{N} \left(\frac{1}{x+n} \right).$$

Then

$$g_N(x+1) = \sum_{n=-N}^{N} \left(\frac{1}{x+1+n} \right) = \sum_{n=-N+1}^{N+1} \left(\frac{1}{x+n} \right) = g_{N-1}(x) + \frac{1}{x+N} + \frac{1}{x+N+1}.$$

Therefore $g(x+1) = \lim_{N\to\infty} g_N(x+1) = \lim_{N\to\infty} g_{N-1}(x) = g(x)$.

C) The functions $f(x)$ and $g(x)$ are odd functions.

For the function f we get

$$-f(x) = -\pi \cot(\pi x) = -\pi \frac{\cos(\pi x)}{\sin(\pi x)} = \pi \frac{\cos(\pi(-x))}{\sin(\pi(-x))} = f(-x).$$

The function g is also odd, because $g_N(-x) = -g_N(x)$.

D) The functions $f(x)$ and $g(x)$ satisfy the functional equation $f(\frac{x}{2})+f(\frac{x+1}{2})=2f(x)$ and $g(\frac{x}{2})+g(\frac{x+1}{2})=2g(x)$.

The function $f(x)$ satisfies the functional expression because of the addition theorems for sine and cosine:

$$f\left(\frac{x}{2}\right)+f\left(\frac{x+1}{2}\right)=\pi\left[\frac{\cos(\frac{\pi x}{2})}{\sin(\frac{\pi x}{2})}-\frac{\sin(\frac{\pi x}{2})}{\cos(\frac{\pi x}{2})}\right]=2\pi\frac{\cos(\frac{\pi x}{2}+\frac{\pi x}{2})}{\sin(\frac{\pi x}{2}+\frac{\pi x}{2})}=2f(x).$$

As a result of

$$\frac{1}{\frac{x}{2}+n}+\frac{1}{\frac{x+1}{2}+n}=2\left(\frac{1}{x+2n}+\frac{1}{x+2n+1}\right)$$

the functional equation is satisfied for g:

$$g\left(\frac{x}{2}\right)+g\left(\frac{x+1}{2}\right)=2g_{2N}(x)+\frac{2}{x+2N+1}.$$

Therefore, the two functions f and g satisfy the four properties. Lets consider the function

$$h(x):=f(x)-g(x)=\pi\cot(\pi x)-\left(\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{2x}{n^2-x^2}\right)\right).$$

This function is continuous on $\mathbb{R}\backslash\mathbb{Z}$ and satisfies the properties **B**, **C** and **D**. As a result of

$$\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x\cos x-\sin x}{x\sin x}\right)=0\Rightarrow\lim_{x\to 0}\left(\pi\cot(\pi x)-\frac{1}{x}\right)=0$$

and

$$\lim_{x\to 0}\sum_{n=1}^{\infty}\left(\frac{2x}{n^2-x^2}\right)=0,$$

the function h is continuously continuable with $\lim_{x\to n}h(x)=0$ for all $n\in\mathbb{Z}$.

Since h is a periodic continuous function, it possesses a maximum m. Let x_0 be a point in the interval $[0,1]$ with $h(x_0)=m$. It follows from property **D** that

$$h\left(\frac{x_0}{2}\right)+h\left(\frac{x_0+1}{2}\right)=2h(x_0)=2m,$$

and hence that $h(\frac{x_0}{2})=m$. Iteration gives $h(\frac{x_0}{2^n})=m$ for all $n\in\mathbb{Z}$ and therefore $h(0)=m$ by continuity. But $h(0)=0$ and so $m=0$, that is $h(x)\le 0$ for all $x\in\mathbb{R}$. As h is an odd function, $h(x)<0$ is impossible. Therefore $h(x)=0$ for all $x\in\mathbb{R}$.

2 Complex Logarithm Function

Analog to the real definition, the complex logarithm of z is a complex number $w = |w| \cdot e^{i\varphi}$, such that $e^w = z$. Since the equation $e^{2k\pi i} = 1$, $k \in \mathbb{Z}$ holds, every nonzero complex number z has infinitely many logarithms. Therefore, for any logarithm w of z, also $w' = w + 2k\pi i$ is a logarithm of z. To assure uniqueness, we define the principal value $\log(z)$ with $\arg(z) \in (-\pi, \pi]$ and $\log(z) := \ln|z| + i \cdot \arg z$.

$\log(z)$ is discontinuous at each negative real number, but continuous everywhere else in $\mathbb{C}^*$. To obtain a continuous logarithm defined on complex numbers, we define the branches of the logarithm. The logarithms w of z of the k-th branch hold the equation

$$w = \ln|z| + i \cdot (\arg z + 2k\pi) \quad k \in \mathbb{Z}.$$

The identities that hold for the "noncomplex" logarithm don't have to hold for the complex logarithm, i.e:

$$\log(-1) + \log(-1) = 2\pi i \neq 0 = \log 1 = \log((-1) \cdot (-1)).$$

3 Application of the Cotangent Function

Problem. For every complex number $z \notin \{0, 1\}$ define

$$f(z) := \sum (\log z)^{-4}$$

where the sum is over all branches of the complex logarithm. Show that there are two polynomials P and Q such that $f(z) = P(z)/Q(z)$ for all $z \in \mathbb{C}\backslash\{0, 1\}$ and show that for all $z \in \mathbb{C}\backslash\{0, 1\}$

$$f(z) = z\frac{z^2 + 4z + 1}{6(z-1)^4}.$$

Solution. From the well-known series for the cotangent function,

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{w + 2\pi i \cdot k} = \frac{i}{2} \cot \frac{iw}{2}$$

and

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{\log z + 2\pi i \cdot k} = \frac{i}{2} \cot \frac{i \log z}{2} = \frac{i}{2} \cdot i \frac{\exp(2i \cdot \frac{i \log z}{2}) + 1}{\exp(2i \cdot \frac{i \log z}{2}) - 1} = \frac{1}{2} + \frac{1}{z-1}.$$

Taking derivatives we obtain

$$\begin{aligned}
\sum \frac{1}{(\log z)^2} &= -z \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2} + \frac{1}{z-1}\right) = \frac{z}{(z-1)^2}, \\
\sum \frac{1}{(\log z)^3} &= -\frac{z}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{z}{(z-1)^2}\right) = \frac{z(z+1)}{2(z-1)^3} \\
\sum \frac{1}{(\log z)^4} &= -\frac{z}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{z(z+1)}{2(z-1)^3}\right) = \frac{z(z^2+4z+1)}{6(z-1)^4}.
\end{aligned}$$