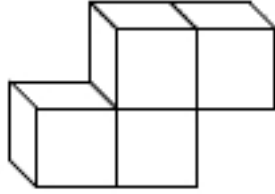


# 1 IMC-Seminar: Exam 1

**Problem 1:** Can a cube of the size  $4 \times 4 \times 4$  be built out of copies of this piece? (This piece consists of 4 unit-cubes while the target-cube consists of  $4^3 = 64$  unit-cubes.)



**Problem 2:** In a convex quadrilateral  $ABCD$ , the angles satisfy  $\angle ABC \geq 120^\circ$  and  $\angle BCD \geq 120^\circ$ . Prove that  $|AC| + |BD| \geq |AB| + |BC| + |CD|$ . (With  $|XY|$  we understand the length of the segment  $XY$ ).

**Problem 3:** Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have pairwise different colors.

**Problem 4:** An  $m \times n$  checkerboard is colored randomly: each square is independently assigned red or black with probability  $1/2$ . We say that two squares,  $p$  and  $q$ , are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at  $p$  and ending at  $q$ , in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than  $mn/8$ .

**Problem 5:** Given a finite string  $S$  of symbols  $X$  and  $O$ , we denote  $\Delta(S)$  as the number of  $X$ 's in  $S$  minus the number of  $O$ 's. (For example,  $\Delta(XOOXOOX) = -1$ .) We call a string  $S$  balanced if every sub-string  $T$  of  $S$  (consecutive symbols) has the property  $-1 \leq \Delta(T) \leq 2$ . (Thus  $XOOXOOX$  is not balanced, since it contains the sub-string  $OOXOO$  whose  $\Delta$  value is -3.)

Find, with proof, the number of balanced strings of length  $n$ .

**Rules:** *You have five hours to work on these problems. No literature/technical devices are allowed.*