Games

Explaining the whole world of Game Theory could take a whole course of studies, so we picked five and a half particular principles that will actually help us solve some of the IMC problems.

Applications: Any problem, where there is a set of *legal moves,* you can choose a *strategy* from, in order to achieve a certain goal.

Terms:

Winning Strategy: A strategy that leads to victory no matter what the opponent does.

Winning Condition: A condition such that you can apply a winning strategy.

Many (but far from all) games we will encounter are *Nim Games* that have the following properties:

- Two Players A and B take turns. A begins.
- Otherwise rules are the same for both of them.
- The first player that can't make a legal move, loses the game.
- One player will lose in a finite number of turns (no loops or draws).

We start with a first example to get into the mood:

GAME 1: A and B alternately place white and black bishops on the squares of a chessboard, which are free and not threatened by an enemy bishop. The loser is the one who cannot move any more. Find a winning strategy for one of the players.

Solution: Considering the vertical symmetry line of the board, B can win by always placing his bishop symmetrically to the bishop previously played by A.

This solution leads us right to the first usefull principle:

Symmetry

Many games can be solved by dividing the set of legal moves into pairs, such that whatever move your opponent makes, there is a subsequent legal move you can make. Hence you win, since your opponent runs out of moves first.

GAME 2: A and B alternately draw diagonals of a regular $2n$-gon that do not intersect any earlier diagonals. The first Player that cannot move, loses the game. Who can force a win?

Solution: A can create a symmetric situation by drawing a main diagonal on his first move. Then after each move of B, he draws the same diagonal reflected at mentioned main diagonal. Thus, A wins.

Once you find a proper symmetry the problem is usually pretty much solved. Hence symmetry mostly provides very compact solutions.

Strategy Stealing

GAME 3: *Double Chess* White and Black alternately make two legal moves. Otherwise standard chess rules. Show that there is a strategy for white wich guarantees him at least a tie.

Solution: Suppose black has a winning strategy. In this case white can *steal* that strategy by just moving a knight to some legal position and back to its original position, thus becoming second player. Therefore White can use supposed strategy and must win. This is a contradiction to the Statement that black has a winning strategy, so there cannot be a winning strategy for black. □

Invariance Principle

Just to get started: Here is an example which will help you to grasp the idea of this principle immediately:

GAME 4: Consider the sequence of the first 2n natural numbers 1,2,3, ... ,2n; where n is odd. In each move, you erase two numbers a, b from the sequence and add the number |a-b| to the sequence. Apparently, after 2n-1 moves, only one number remains. Is it even or odd?

Solution: The invariant in this example is the parity of the sum of the numbers. In the beginning, this sum is $S = \sum_{i=1}^{2n} i = \frac{(2n)(2n+1)}{2} = n(2n+1)$, which is odd. In each step, you change S as follows: $S_{new} = S - a - b + |a - b|$. Now we consider three cases:

a,b are odd: -a-b is even |a-b| is even, hence -a-b+|a-b| is even.

a,b both even: -a-b+|a-b| is even.

a odd, b even: -a-b is odd and |a-b| is odd, hence -a-b+|a-b| is even. The same for b odd and a even.

Therefore, S remains odd in each turn, hence an odd number will remain in the end. □

Now let us think about the invariance principle more generally. The goal always is to find an object or a function (or something else), which remains invariant. Generally, remember this saying:

If there is a repetition, look for what does not change!

To succeed in finding an invariant, you might ask yourself:

What changes?

What have all objects in common?

You will also find it helpful, to keep the following list in mind, whenever you have a starting state S and a sequence of legal steps:

1. Can a given end state be reached?
2. Find all reachable end states!
3. Is there a convergence to an end state?
4. If there is no convergence, what can you say about periods?

Remember this principle, it is usually very helpful, whenever games, algorithms or transformations are concerned.

To illustrate this even more, here is another example:

GAME 5 (Würfl's game): Alice and Bob play a game. They have a board with horizontal lines on it and a finite amount of stones placed in any order on the lines. Alice and Bob move alternately, Alice can always choose any subset T of the set S of stones on the board. Then Bob in his turn can choose either T or $\overline{T}$; then he removes all stones from the chosen set from the board and puts each stone from the remaining set one line up. Alice' goal is to get one stone to the top line, then she wins. Bob wins if no stones remain. Find winning conditions if both player play optimal.

Hints: What changes? - positions and amout of stones.

What have the stones and lines to do with each other?

Can you give each stone a weight, such that you can define an invariant?

Solution: Start with the top line: give each stone on the i-th line the weight $\frac{1}{2^{i-1}}$, a removed stone shall get weight 0. Define N as follows: $N_i = \sum_{s \in S} f(s)$, where f is the weight function. This gives a weighted sum after the i-th turn. Now work backwards (which is another very useful principle): If Alice wins in the - say j-th turn - apperently $N_j \geq 1$. If Alice plays wise, the inequality $N_i \geq 1$remains invariant:

If $N_0 \geq 1$in the beginning, Alice can choose any subset T, whose weighted sum $M_i = \sum_{s \in T} f(s) = \frac{1}{2}$ in each of her turns. In this case, both of her sets T and $\overline{T}$ have a weighted sum $M_i = \sum_{s \in T} f(s) = \frac{1}{2}$ and $\overline{M_i} := \sum_{s \in \overline{T}} f(s) \geq \frac{1}{2}$. Therefore, it doesn't matter which subset Bob choses, he will be faced with a subset whose weighted sum is≥ 1, since the weigth of every stone doubles if moved one line up. Therefore Alice wins.

What happens if $N_0 < 1$? Here Bob can always choose a subset T in his i-th move, whose weighted sum is $\geq \frac{1}{2} N_{i-1}$. Hence Alice will be faced with a set of stones, whose total sum is $< N_i < 1$. Therefore, she can't reach her goal, because it would require a weighted sum ≥ 1.

Therefore the winning conditions are:

If $N_0 < 1$Bob wins, if $N_0 \geq 1$ Alice wins. $\square$

This much about invariances. Another very useful idea is to look for something related to the invariance prinsiple: variants.

Since this strategy is a very closely related topic to invariances, we shall only give the basic ideas of this principle. If you want to train it, there are 2 training examples where you can apply that approach. The idea is to define an object, a function, ... that changes in every turn or over a series of turns. If you can prove that it will always change in the same "way", you will end up in a final state if you have a constraint that permits any further move.

E.g. you have a series of positive numbers, which shall remain positive in each turn. If you can show, that it is decreasing strictly each turn by at least one certain positive real number, you will reach a state, where no legal move is possible any more, since the series only consists of positive numbers.

L-W-Partitioning

A very important approach to Nim Games is L-W-Partitioning.

Thereby we think of it as a directed graph, where each position is represented by a vertex and directed edges depict the moves.

We now partition the set P of all positions into a set L of losing positions an a set W of winning positions: $P = L \dot{\cup} W$

A player facing a position in L at the beginning of his or her turn will loose given a clever opponent.

An apt player starting their turn in a winning position can win no matter what their opponent does.

Thus, from every position in W there must be a legal move to a position in L and every move from a position in L must result in a position in W.

Any final position from which there is no move out obviously belongs to L.

GAME 6: *Batchet's Game* There are n stones on the table. A and B alternately remove between 1 and k stones. The winner is the one to take the last stone. Find a winning condition for A.

Hint: Observe k=4, n=10 and n=11.

Solution: Exemplary: k=4 We begin at the end, looking at 0, which is in L. Hence 1, 2, 3, 4 are in W. It follows that 5 is in L, since every legal move from 5 leads to W$(1, 2, 3, 4 \in W)$. Further on:

$$\underbrace{0}_{L}\underbrace{1\,2\,3\,4}_{W}\underbrace{5}_{L}\underbrace{6\,7\,8\,9}_{W}\underbrace{10}_{L}\underbrace{11}_{W}$$

It becomes evident, that the set L consists of all multiples of 5. Hence n=10 leads to a loss of player A, while n=11 leads to his victory.

For arbitrary k, respectively the set L consists of any multiple of k+1. Therefore the winning condition for A is that n is not a multiple of k+1.

The previous example evinces a strategy:

1. Look at easy exemplary cases.
2. Work backwards, starting at final positions.
3. Look at the first few winning and loosing positions.
4. Try to find a pattern.

Another example to deepen the use of L-W-Partitioning:

GAME 7: A and B play a game. Initially, there are 10^7 stones on a table. In each turn, p^n stones can be removed, where p is any prime and $n \in \mathbb{N}$. The Player who takes the last stone wins. Who wins?

Solution: We work backwards again and observe that 6 is the first number that is not the power of a prime and thus the first losing position. You cannot get from one multiple of 6 to another, but can attain a multiple of 6 from each number that is not a multiple of 6 by substracting a number between 1 and 5. Hence L consists of all multiples of 6.

The Extremal Principle

This principle is of truly universal applicability. This principle tells you to choose an (in some way) extremal object.

Here is an example to help you to get the idea.

GAME 8: Every road in Bavaria is one way. Every pair of cities is connected exactly via one road. (So you can think of a directed finite graph, where every two distinct vertices have one edge.) Show that there is a city that can be reached from every other city directly or via at most one other city.

Hints: What objects do you have, in which way are the objects different?

Can you take an object with a maximal property?

Solution: Extremal principle:

Take a city M of the set of cities with the most incoming roads. Claim: This is a city with the property we want.

First, partition the set C of cities as follows: $C = \{M\}\dot{\cup}D\dot{\cup}E$ where D is the set of cities, which have direct roads to M. Then M has |D| incoming roads.

Now consider two cases: The first one, |E|=0 is trivial.

Now let |E|>0. Then, there exists an $A \in E$. Take such an arbitrary A. Now I claim that $\exists X \in D$, such that there is a direct road from A to X (which has a road to M). Consider the converse:

If such an X would not exist, there would be |D| direct roads to A from D, but also one from the city M, since A has no direct road to M. Hence, there would be at least |D| + 1 direct roads to A, which contradicts the maximal choice of M. □

By solving this example, we see very well, what is important considering extremal elements. Most times an extremal element is chosen first, then the statement is proven by contradiction by showing that any slight variation would increase/decrease your extremal property or contradict the special choice.

Another interesting point is, that the extremal prinicple is usually constructive since you make a choice of objects.

The math behind this principle is indeed simple, but it is worth remembering it.

1. Any finite set of real numbers has a Maximum and a Minimum.
2. Any set of integers, which is bounded below, has a Minimum.
3. Any bounded set of real numbers has an Infimum and a Supremum.

Most times, the extremal principle does not require more (set-) theory.

Now, at last, we present a variation of the extremal principle:

It is called *Minimize-Maximize*. Most times this strategy is used to show that a choice/an object is optimal. Very often this is proved by showing that a property can be reached at most and at least at the same time. Here is a game illustrating it.

GAME 9: Angela and Ben play a game. In front of them there is the sequence 0,1 , 2, ... , 2^{2n} written on a paper, where n is any positive integer. Both players move alternately. Angela starts. In her first turn, Angela erases 2^{2n-1}numbers, then Ben erases 2^{2n-2}, then Angela 2^{2n-3} and so on until B erases $2^0 = 1$ number. Since $\sum_{i=0}^{2n-1} 2^i = \frac{2^{2n}-1}{2-1} = 2^{2n} - 1$, only two numbers a, b will remain in the end. Angela wins |a-b| from Ben. How should Angela play to win as much as possible and how should Ben play to lose as little as possible? How much does Angela win if both apply their optimal strategy?

Hints: Consider a strategy for each player.

Try to define algorithms.

Show that the algorithms give a lower boundary for the loss and an upper boundary for the win respecticely.

Solution: Angela wins 2^n. We show this by applying a minimize-maximize-strategy.

First we find a strategy which guarantees, that Angela won't win less, regardless of what Ben does: In each of her turns, Angela erases every second remaining number, which means she increases the difference of two neighbouring numbers. Therefore, in her first turn, she erases 1, 3, 5, ..., $2^{2n} - 1$. Hence, the the difference between two numbers is at least 2. If she repeats this procedure, in each of her turns, she at least doubles the difference between two neighbouring numbers. So, as she has n turns, the final two numbers will have a difference of at least 2^n, meaning she has a lower constraint of $2^n$for her win.

We conclude Angela's optimal strategy: Erasing every second remaining number.

Now we give a strategy for Ben, guaranteeing, that Angela won't win more whatever she does. We see such a strategy by observing the following: Angela always moves before B does and Angela has already erased half of the numbers in her first turn, which means there are at most half of the numbers $> \frac{2^{2n}}{2} = 2^{2n-1}$or $< \frac{2^{2n}}{2} = 2^{2n-1}$ remaining. Therefore Ben can cross out 2^{2n-2}consecutive numbers in the "half", where one of the inequalities holds in his first turn, minimizing the maximal absolute value to 2^{2n-1}. Ben repeates this process by erasing consecutive numbers at the beginning or the end in each of his turns. Hence, he minimizes the maximum of the absolute value of the remaining integers to 2^{2n-i}after his i-th turn. Since he has n turns, he reduces the maximal possible absolute value to at least $2^{2n-n} = 2^n$. Angela will never win more than 2^n.

Ben's optimal strategy: erasing consecutive numbers as described above.

Therefore Angela will win 2^n(let us say €), if she and Ben apply their optimal strategy. □

With this example, we have finished our presentation of (game) solving strategies. Of course there are more strategies worth considering like *colouring*, *working backwards*, etc. we had no time for. If you have further interest, may have a look into the following literature.

Literature:

Engel, Arthur: "*Problem Solving Strategies*", Springer, 1998, chapters 1, 2, 3, 13

Grinberg, Natalia: "*Lösungsstrategien - Mathematik für Nachdenker*", Harri Deutsch, first edition, 2008, chapters 3, 4

Polya, George: "*How to Solve it - a new aspect of the mathematical method*", Princeton University Press, 2004

Examples taken from above literature and the following <u>competitions</u>:
Tournament of the Towns: http://www.math.toronto.edu/oz/turgor/archives.php
William Lowell Putnam Mathematics Competition:
http://www.amc8.org/a-activities/a7-problems/putnamindex.shtml
International Mathematical Olympiad:
http://www.imo-official.org/problems.aspx?&column=year&order=desc&language=en

Further Examples:

EASY:

EXAMPLE 1: Start with the set {3,4,12}. In each step you may remove any two numbers and replace them with 0.6a-0.8b and 0.8a+0.6b respectively. Can you reach the goal {4,6,12} in finitely many steps? Can you reach the goal {x,y,z} with |x-4|, |y-6|, |z-12| each less than $\frac{1}{\sqrt{3}}$?

EXAMPLE 2: Assume an 8×8-chessboard with usual colouring. You may repaint all squares in a column of one 2×2 block. Can you reach just one black square?

EXAMPLE 3: You have a matrix which is filled with positive integers. In one move, you can either subtract 1 from each element of a column or double each element of a row. Can you reach the zero matrix? If you can, prove it.

EXAMPLE 4: Starting with $n = 2$ two players A and B move alternately by adding a proper divisor of n to the current n. The winner is the first player to surpass 1990. Who wins?

EXAMPLE 5: A and B alternately write positive integers $\leq p$ on the blackboard, where divisors of numbers already on the board are forbidden. The one who cannot move anymore loses. Who wins for *(a)* $p = 10$? *(b)* $p = 1000$?

HARD:

EXAMPLE 6: Consider lattice squares {x,y} with x,y positive integers. Assign to each it's lower left corner as a label. Shade the squares (0,0), (0,1), (1,0), (1,1), (0,2), (2,0).There is a chip on each square. If (x,y) is occupied, but (x+1,y) and (x,y+1) are free, remove the chip from (x,y) and place chips on (x+1,y) and (x,y+1). Is it possible to remove all chips from the shaded squares?

EXAMPLE 7: Of 2n+3 points on a plane, no three are collinear and no four lie on a circle. Show that you can choose 3 points such that the circle through them partitions the remaining 2n points into two subsets, each with n points.

EXAMPLE 8: 100 students attend a lecture. Each student comes and goes as he likes (but only once respectively). You know that among every three arbitrarily chosen students, 2 of them have been together in the lecture for some time. The professor wants to announce something important everyone should hear. How often at least and when does he have to say it, so that everyone hears it?

EXAMPLE 9: A power grid has the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one test technician can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Technician knows that current will circulate from any node to any other node. What is the least number of tests which is required to demonstrate this?

EXAMPLE 10: *Wythoff's Game:* There are two piles of stones on a table. A and B alternately either take any number of stones from one pile or the same number of stones from each pile. The winner is the one to take the last stone. Find an algorithm to calculate all losing positions.

EXAMPLE 11: A and B move alternately. From a pile of n stones, A removes $t \in \{1, 2, ..., n-1\}$ stones at his first move. From then on, a player may remove any number that is a divisor of the number of stones taken at the preceding move. The winner is the one to make the last move. Wich initial positions are winning for A or B?

EXAMPLE 12: Albert and Bertram play a game. In front of them, there lie 2008 vectors on a table. They move alternately, Albert starts. In each move, one player selects one vector. After the 2008 moves, they calculate the vector sum of their vectors and compare the length of it. The player with the shorter length loses. (Note: A draw is possible) Is there a strategy for Albert not to lose?

EXAMPLE 13: 2n professors are invited to a banquet. Each professor has at most n-1 enemies among the other professors. (Here hostility is a symmetric relation - yes you can think of it as a graph). Can you find a seating plan (seat them around a circular table), such that each professor does not seat next to one of his enemies?

VERY HARD:

EXAMPLE 14: The game is played on an infinite chessboard. An $n \times n$ block is filled with chips. In each move, you may use one chip to jump over one other chip onto a free cell verticaly or horizontaly. The chip you jumped over is then removed. The goal is to end up with one chip in the end. Find all n for which you can do this.

HINTS:

Ex 1. Invariance

Consider the set as a point in space.

Consider the distance to the origin.

Ex 2. Invariance

Consider parity.

Ex 3. Variance

Work backwards.

Find a series of turns that decreases something.

Ex 4. L-W-Partitioning

Look at even and odd numbers.

Ex 5. (a) Symmetry

(b) non constructive

Ex 6. Invariance

Work backwards.

Define a weight function similarly to the weight function in Würfl's game using x, y and the lattice numeration.

<u>*Ex 7.*</u> Extremal principle

Try to define a list, where you can choose a maximal element.

Choose A, B so that all points lie on the same side of the line passing through A and B.

Remember the peripheral angle theorem.

Arrange the points according to the angle between A, the point P and B.

<u>*Ex 8.*</u> Extremal Principle

Define a time interval and for each student a subset of it.

Think of the Minimum and Maximum (first and last student).

<u>*Ex 9.*</u> Symmetry

Look at a main diagonal.

<u>*Ex 10.*</u> L-W-Partitioning

Work backwards.

Observe the difference between the numbers of stones on each pile.

<u>*Ex 11.*</u> Symmetry

Use Powers of 2.

How can A or B create a symmetric situation?

<u>*Ex 12.*</u> Extremal Principle

Find a clever way to compare the vectors.

Calculate the vector sum of all 2008 vectors and the projection of each vector on it.

Choose the maximal element. Prove that this strategy works.

<u>*Ex 13.*</u> Variance

Seat them in any order and try to improve their arrangement.

Can you give an algorithm that decreases the number of hostile couples seating next to each other.

<u>*Ex 14.*</u> Invariance

Think of a clever pattern of chips you can reduce easily to one.

The above pattern is not quadratic.

Can you go "round" the $n \times n$ block with the pattern?

What are you left over with?

Consider n modulo 3.