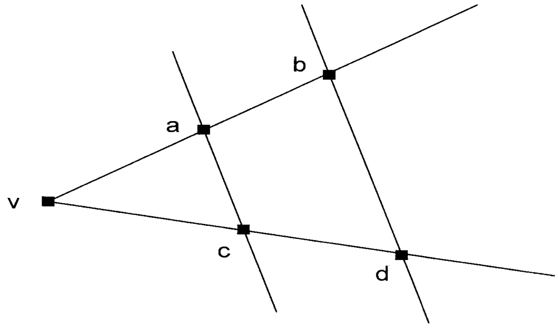
Geometry

Toolbox for solving geometry problems:

- <u>Intercept theorem</u>

  If you have two parallel lines intercepted by two other non-parallel lines, then:

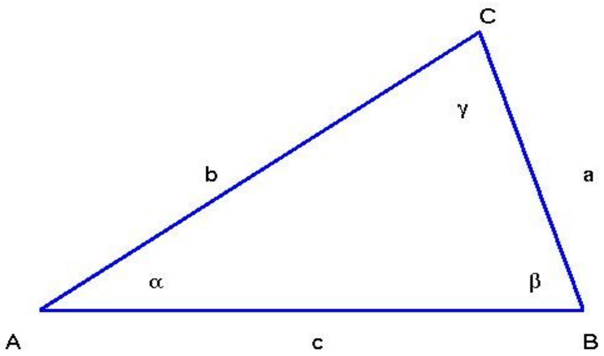
  $$\frac{vb}{bd} = \frac{va}{ac}$$



- <u>Law of sines</u>

  In a triangle with sides a,b,c and respective oppsite angles α,β,γ we have:

  $$\frac{a}{\sin(\alpha)}=\frac{b}{\sin(\beta)}=\frac{c}{\sin(\gamma)}$$



- <u>Addition theorems</u>

  sin(x+y) = sin(x) cos(y) + sin(y) cos(x)
  cos(x+y) = cos(x) cos(y) - sin(x) sin(y)

- <u>„Finding the right line“</u>

  If we search a point which has to fulfill some geometrical conditions, we can translate these geometrical conditions into lines (or other curves) and then intersect them for finding the interesting piont.

- <u>Other theorems like the pythagorean theorem or the thales' theorem</u>

Proposition:

Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ linear, $M \subseteq \mathbb{R}^n$. Then $Vol(T(M)) = |det\, T| \cdot Vol(M)$

Task: (IMC 2009)

Let ℓ be a line and P a point in R^3. Let S be the set of points X such that the distance from X to ℓ is greater than or equal to two times the distance between X and P. If the distance from P to ℓ is d > 0, find the volume of S.

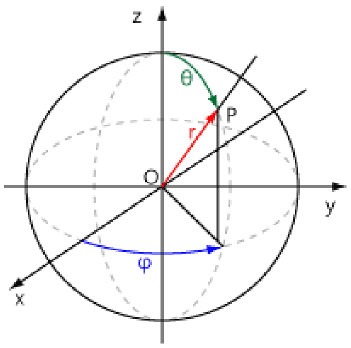
Choosing the right coordinate system

Choosing the right coordinate system is important for reducing the complexity of the problem. Here are two different coordinate systems:

Spherical coordinate system

cartesian coordinates => spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \arccos\left(\frac{z}{r}\right) \qquad \varphi = \arctan\left(\frac{y}{x}\right)$$



spherical coordinates => cartesian coordinates:

$$x = r \cdot \cos(\theta) \cdot \sin(\varphi) \qquad y = r \cdot \sin(\theta) \cdot \sin(\varphi) \qquad z = r \cdot \cos(\theta)$$

Cylindrical coordinate system

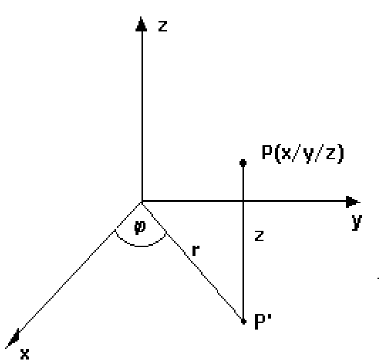
cylindrical coordinates => cartesian coordinates:

$$x = r \cdot \cos(\varphi) \qquad y = r \cdot \sin(\varphi) \qquad z = z$$

cartesian coordinates => cylindrical coordinates:

$$r = \sqrt{(x^2 + y^2)} \qquad z = z$$

$$\varphi = \begin{cases} 0 & for\ x = 0, y = 0 \\ \arcsin\left(\frac{y}{r}\right) & for\ x \geq 0 \\ -\arcsin\left(\frac{y}{r}\right) + \pi & for\ x < 0 \end{cases}$$



Duality

Take a proposition and interchange the following words, the proposition will remain true:

point	⇔	line	
connection line	⇔	intersection	
3 points are collinear	⇔	3 lines are concurrent	(intersect in one piont or parallel)

<u>Example:</u> Pappus' theorem, Desargues' theorem

Homework

<u>Putnam 2006 A1</u>

Find the volume of the region of points (x, y, z) such that $(x^2+y^2+z^2+8)^2 \leqslant 36(x^2+y^2)$.

<u>Putnam 2008 B3</u>

What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

<u>Putnam 1998 B3</u>

Let H be the unit hemisphere $\{(x, y, z): x^2+y^2+z^2=1, z\geqslant 0\}$,
C the unit circle $\{(x, y, 0): x^2+y^2=1\}$, and P the regular pentagon inscribed in C.
Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form A sin (α) + B cos (β), where A,B, α,β are real numbers.

<u>Putnam 2005 A6</u>

Let n be given, $n\geqslant 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are P_i . What is the probability that at least one of the vertex angles of this polygon is acute?

<u>Putnam 2008 B1</u>

What is the maximum number of rational points that can lie on a circle in R2 whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)