Induction

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The following script is intended as a short review of general induction principles for the competition IMC. As the theory of induction should be well-known, this script focuses on common types of induction problems.

The IMC has decreased problems which can be solved through induction gradually over the years. Therefore, this script should be regarded as a short review of basic principles instead of a exhaustive preparation for the competition.

1 Theory

The first known complete induction can be found in Francesco Maurolico's Arithmeticorum Libri Duo (1575), where this technique is used to prove that the sum of the first n odd integers is n^2. The first explicit formulation of the principle of induction was given by Pascal in his *Traité du triangle arithmétique* (1665). It got popular when Gottlob Frege defined the natural numbers with the aid of this method.

The technique of mathematical induction is based on the 5-th Peano-Axiom for the set of natural numbers: Let K be a subset of $\mathbb{N}$ with the following attribute. 1 is a subset of $\mathbb{N}$ and if $k \in K$, then $k + 1 \in K$.

In order to prove an equation $p(n) = q(n)$ for any natural number the equation has to be proven for the case $n = 1$, thus for at least one $n = k$. If the equation is also valid for $n = k + 1$, it is valid for any natural number.

There are various different variations of induction. The most common ones are listed below.

1. Use of multiple predecessors. Sometimes the equation depends on more than one predecessor. In this case, the equation has to hold for $n = k$ and $n = k - 1$ (and so on) to prove that it holds for $n = k + 1$.
2. Induction on more than one counter. This method is used to prove a statement involving two natural numbers n and m. The basis and the inductive step have to be performerd for n and m.
3. Forward-Backward-Induction. Multiple inductive Steps are used. For example, $n \to 2n$ is shown in the first step and $n \to n-1$ in the second step in order to close the gaps.

2 Examples

Problem 1. There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own. If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness.

All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople). One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe. One evening, he addresses the entire tribe to thank them for their hospitality. However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

What effect, if anything, does this faux pas have on the tribe?

Problem 2. Let n be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \qquad b_k = 2^{k-n}, \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \ldots + \frac{a_n - b_n}{n} = 0.$$

Problem 3. Let X be a set of $\binom{2k-4}{k-2} + 1$ real numbers, $k \geq 2$. Prove that there exists a monotone sequence $\{x_i\}_{i=1}^{k} \subseteq X$ such that

$$|x_{i+1} - x_1| \geq 2|x_i - x_1|$$

for all $i = 2, \ldots, k-1$.

Problem 4. Prove the following inequation through induction.

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 \qquad \forall n \in \mathbb{N}$$

Problem 5. Let $P(x) = x^2 - 1$. How many distinct real solutions does the following equation have:

$$\underbrace{P(P(\ldots(P}_{2011}(x))\ldots))?$$

3 Exercises

Exercise 1. Let $f(x) = 2x(1-x)$, $x \in \mathbb{R}$. Define

$$f_n := \underbrace{f \circ \ldots \circ f}_{n}.$$

Compute $\int_0^1 f_n(x)dx$ for $n = 1, 2, \ldots$.

Exercise 2. Let V be a real vector space, and let $f, f_1, f_2, \ldots, f_k$ be linear maps from V to $\mathbb{R}$. Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \ldots = f_k(x) = 0$. Prove that f is a linear combination of $f_1, f_2, \ldots, f_k$.

Exercise 3. For each $n \geq 1$ let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \qquad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n \cdot b_n$ is an integer.

Exercise 4. Let $a_1 = 1$, $a_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$. Show that

$$2/3 \leq \limsup_{n \to \infty} |a_n|^{1/n} < 2^{-1/2}.$$

Exercise 5. Let n be a positive integer. An n-simplex in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \ldots, P_n$, called its vertices, which do not all belong to the same hyperplane. For every n-simplex S we denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S. Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \cdots + v(S_n)C(S_n) = v(S)C(S).$$

Exercise 6. A judge tells a condemned prisoner that he will be hanged at noon on one weekday in the following week but that the execution will be a surprise to the prisoner. If the prisoner is not surprised by the date of the execution, he will not be executed. However, he will not know the day of the hanging until the executioner knocks on his cell door at noon that day. Can the prisoner induce the date of the execution and save his life?

Exercise 7. Let A be an $n \times n$ complex matrix such that $A = \lambda I$ for all $\lambda \in C$. Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

4 Appendix

Solution 1. Let n be the number of blue-eyed islanders. When $n = 1$ the single blue-eyed person realizes that the traveler is referring to him or her, and thus commits suicide on the next day.

Now suppose $n > 1$. Each blue-eyed person sees $n - 1$ blue-eyed persons and will reason as follows: "If I am not blue-eyed, then there will only be $n - 1$ blue-eyed people on this island, and so they will all commit suicide $n - 1$ days after the traveler's address". But when $n - 1$ days pass, none of the blue-eyed people do so (because at that stage they have no evidence that they themselves are blue-eyed). After nobody commits suicide on the $(n - 1^{\text{st}})$ day, each of the blue-eyed people then realizes that they themselves must have blue eyes, and will then commit suicide on the n-th day. □

Solution 2. Since $k\binom{n}{k} = n\binom{n-1}{k-1}$ for all $k \geq 1$, the problem is equivalent to

$$\frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right] = \frac{2^1}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n}.$$

We prove this equation by induction. For $n = 1$, both sides are equal to 2. Assume that the equation holds for some n. Let

$$x_n = \frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right];$$

then

$$\begin{aligned} x_{n+1} &= \frac{2^{n+1}}{n+1}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = \frac{2^n}{n+1}\left(1 + \sum_{k=0}^{n-1}\left(\frac{1}{\binom{n}{k}} + \frac{1}{\binom{n}{k+1}}\right) + 1\right) \\ &= \frac{2^n}{n+1}\sum_{k=0}^{n-1}\frac{\frac{n-k}{n} + \frac{k+1}{n}}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = \frac{2^n}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = x_n + \frac{2^{n+1}}{n+1}. \end{aligned}$$

This implies the equation for $n + 1$. □

Solution 3. In order to prove the statement, we generate a more general statement (which is our lemma). Let $k, l \geq 2$, let X be a set of $\binom{k+l-4}{k-2} + 1$ real numbers. Then either X contains an increasing sequence $\{x_i\}_{i=1}^{k} \subseteq X$ of length k and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \qquad \forall i = 2, \ldots, k - 1,$$

or X contains a decreasing sequence $\{x_i\}_{i=1}^{l} \subseteq X$ of length l and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \qquad \forall i = 2, \ldots, l - 1.$$

We use induction on $k + l$. In case $k = 2$ or $l = 2$ the lemma is obviously true. Now let us make the induction step. Let m be the minimal element of X, M be its maximal element. Let

$$X_m = \{x \in X : x \leq \frac{m + M}{2}\}, X_M = \{x \in X : x > \frac{m + M}{2}\}.$$

Since $\binom{k+l-4}{k-2} = \binom{k+(l-1)-4}{k-2} + \binom{(k-1)+l-4}{(k-1)-2}$, we can see either

$$|X_m| \geq \binom{(k-1)+l-4}{(k-1)-2} + 1, \qquad \text{or} \qquad |X_M| \geq \binom{k+(l-1)-4}{k-2} + 1.$$

In the first case we apply the inductive assumption to X_m and either obtain a decreasing sequence of length l with the required properties (in this case the inductive step is made), or obtain an increasing sequence $\{x_i\}_{i=1}^{k-1} \subset X_m$ of length $k-1$. Then we note that the sequence $\{x_1, x_2, \dots, x_{k-1}, M\} \subseteq X$ has length k and all the required properties.
In the case $|X_M| \geq \binom{k+(l-1)-4}{k-2} + 1$ the inductive step is made in a similar way. Thus the lemma is proved.
The reader may check that the number $\binom{k+l-4+1}{k-2} + 1$ cannot be smaller in the lemma.□

Solution 4. In order to prove the statement, we generate a more general statement. The inequation

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n} \qquad \forall n \in \mathbb{N}$$

can be proven through a simple induction and implies the original problem. □

Solution 5. Put $P_n(x) = \underbrace{P(P(\dots(P}_{n}(x))\dots))$. As $P_1(x) \geq -1$, for each $x \in R$, it must be that $P_{n+1}(x) = P_1(P_n(x)) \geq -1$, for each $n \in N$ and each $x \in R$. Therefore the equation $P_n(x) = a$, where $a < -1$ has no real solutions. Let us prove that the equation $P_n(x) = a$, where $a > 0$, has exactly two distinct real solutions. To this end we use mathematical induction by n. If $n = 1$ the assertion follows directly. Assuming that the assertion holds for a $n \in N$ we prove that it must also hold for $n+1$. Since $P_{n+1}(x) = a$ is equivalent to $P_1(P_n(x)) = a$, we conclude that $P_n(x) = \sqrt{a+1}$ or $P_n(x) = -\sqrt{a+1}$. The equation $P_n(x) = \sqrt{a+1}$, as $\sqrt{a+1} > 1$, has exactly two distinct real solutions by the inductive hypothesis, while the equation $P_n(x) = -\sqrt{a+1}$ has no real solutions (because $-\sqrt{a+1} < -1$). Hence the equation $P_{n+1}(x) = a$, has exactly two distinct real solutions.

Let us prove now that the equation $P_n(x) = 0$ has exactly $n+1$ distinct real solutions. Again we use mathematical induction. If $n = 1$ the solutions are $x = \pm 1$, and if $n = 2$ the solutions are $x = 0$ and $x = \pm 2$, so in both cases the number of solutions is equal to $n + 1$. Suppose that the assertion holds for some $n \in N$. Note that $P_{n+2}(x) = P_2(P_n(x)) = P_n(x)(P_n(x) - 2)$, so the set of all real solutions of the equation $P_{n+2} = 0$ is exactly the union of the sets of all real solutions of the equations $P_n(x) = 0$, $P_n(x) = 2$ and $P_n(x) = -2$. By the inductive hypothesis equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions, while the equations $P_n(x) = 2$ and $P_n(x) = -2$ have two and no distinct real solutions, respectively. Hence, the sets above being pairwise disjoint, the equation $P_{n+2}(x) = 0$ has exactly $n + 3$ distinct real solutions. Thus we have proved that, for each $n \in N$, the equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions, so the answer to the question posed in this problem is 2012. □