Rank of a Matrix

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Only few problems dealing with the rank of a given matrix have been posed in former IMC competitions. Most of these problems have quite straightforward solutions, which only use basic properties of the rank of a matrix. Others demand some familiarity with eigenvalues and Jordan normal forms. The script at hand is intended to provide sufficient information on these topics for solving the given problems.

1 Rank of a Matrix

1.1 Definition, Row and Column Operartions, Determinants

The *rank* of a Matrix $A \in M(n \times n, \mathbb{K})$ is defined as the dimension of its column space, which equals the number k of linear independent column vectors. Set $rk(A) := k$.

Proposition: $\forall n, m \in \mathbb{N}, A \in M(n \times m, \mathbb{K}) : rk(A) = rk(A^T)$

Proof: Interpret A as a linear map $A : \mathbb{K}^m \to \mathbb{K}^n$, let $k := rk(A) = \dim Ran(A)$. Using the identity

$$m = \dim ker(A) + \dim Ran(A)$$

we get

$$k = m - \dim ker(A) = m - (m - \dim Ran(A^T)) = \dim Ran(A^T).$$

Exercise 1: (Putnam 2008)
Alan and Barbara play a game in which they take turns filling entries in an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it into a vacant entry. The game ends, when all entries are filled. Alan wins, if the determinant of the resulting matrix is nonzero, otherwise Barbara wins. Who has a winning strategy?

One can solve this task using the fact that the determinant of a matrix is zero if the matrix has not complete rank. However, in most cases this is far from obvious. In order to compute the rank one usually transforms the given matrix into a simpler form using row and column operations of the following kind:

1. Swap the ith row /column with the jth row / column
2. Multiply the ith row / column with some scalar $\lambda \neq 0$
3. Add a multiple of the ith row / column to the jth row / column

Lemma: Row and column operations do not alter the rank of a matrix $A \in M(n \times m, \mathbb{K}$.

Proof: Row and column operations of A are matrix multiplications by an elementary matrix B. B has complete rank, thus AB or BA have the same rank as A.

The following problem can be solved by row and column operations:

Exercise 2: (IMC 2005)
Let A be a $n \times n$ whose $(i,j)^{th}$ entry is $i+j$ for all $i, j = 1, 2, ..., n$. What is the rank of A?

Given the special case of a square matrix, we can use the *determinant* of the matrix to check whether it has complete rank or not.

Definition: Let $\mathbb{K}$ be a field, $n \in \mathbb{N}$. A map

$$\det : M(n \times n; \mathbb{K}) \to \mathbb{K}, \quad A \to \det A$$

is called *determinant*, iff

$$\det \text{is } \textit{alternating multilinear} \text{ with regard to rows} \tag{1}$$
$$\det E_n = 1 \tag{2}$$

Corollary: Properties of $\det : M(n \times n; \mathbb{K}) \to \mathbb{K}$. $\forall A, B \in M(n \times n; \mathbb{K})$:

$$\text{There exists exactly one function with the properties (1) and (2)} \tag{3}$$
$$\forall \lambda \in \mathbb{K} : \det(\lambda \cdot A) = \lambda^n \cdot \det A \tag{4}$$
$$\text{If any row of A is zero, then } \det A = 0 \tag{5}$$
$$\det A = 0 \Leftrightarrow \operatorname{rank} A < n \tag{6}$$
$$\det A \cdot B = \det A \cdot \det B, \text{especially} \det A^{-1} = (\det A)^{-1} \tag{7}$$
$$\text{If } \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} : \det \text{is continuous.} \tag{8}$$

Proof: See Gerd Fischer, *Lineare Algebra, p.178 ff.*

1.2 Inequalities

We list two important inequalties of the rank of a matrix, *Frobenius'* and *Sylvester's* inequality:

Proposition: (Frobenius Inequality). Let $A \in M(k \times n, \mathbb{K})$, $B \in M(n \times m, \mathbb{K})$, $C \in M(m \times l, \mathbb{K})$. Then

$$\operatorname{rank} BC + \operatorname{rank} AB \leq \operatorname{rank} ABC + \operatorname{rank} B$$

Proof: If $U \subset V$ and $X : V \to W$, then

$$\dim (\ker X|_U) \leq \dim \ker X = \dim V - \dim \operatorname{Ran} X.$$

For $U = \operatorname{Ran} BC, V = \operatorname{Ran} B$ and $X = A$ we get

$$\dim (\ker A|_{\operatorname{Ran} BC}) \leq \dim \operatorname{Ran} B - \dim \operatorname{Ran} AB.$$

Thus, $\dim (\ker A|_{\operatorname{Ran} BC}) = \dim \operatorname{Ran} BC - \dim \operatorname{Ran} ABC$.

Proposition: (Sylvester's Inequality). Let $A \in M(k \times n, \mathbb{K})$, $B \in M(n \times m, \mathbb{K})$. Then

$$\operatorname{rank} A + \operatorname{rank} B \leq \operatorname{rank} AB + n,$$

Proof: Make use of Frobenius' inequality for matrices $A = A, B = E_n$ and $C = B$.

Exercise 3: (IMC 2007) Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?

1.3 Eigenvalues and Normal forms

The theory of eigenvalues and normal forms of matrices is a broad theory which is covered in a second semester lecture on linear algebra. That is why we are not aiming at giving a comprehensive introduction on eigenvalues and normalforms, instead we will derive some basic principles while solving an IMC task. We will also present a short solution to the problem using more sophisticated tools like the Jordan normal form.
Consider the following problem:

Exercise 4:(IMC 2000)
Let A and B be square complex matrices of the same size, $\operatorname{rank}(AB - BA) = 1$. Show that $(AB - BA)^2 = 0$.

Proof: For the sake of contradiction, assume that $(AB - BA)^2 \neq 0$. Using a geometrical interpretation, we can conclude that there is a 1 - dimensional subspace $U \subset V$ which remains *invariant* while using the map $(AB-BA) : V \to W$. As U has dimension one, $(AB-BA)$ has to act on V like a scalar multiplication, that is: $\forall u \in U : (AB - BA)(u) = \lambda \cdot u,$ for some constant $\lambda \in \mathbb{C}\backslash\{0\}$. Now we can introduce the concept of an *eigenvalue* of a matrix:

Definition: Let F be a linear map $F : V \to V$ and $\lambda \in \mathbb{K}$. If there exists $v \neq 0$ with $F(v) = \lambda v$, λ is called an *eigenvalue* of F, v an *eigenvector*.

Differently speaking, we have to show that $(AB-BA)$ has only zeros as eigenvalues, as $\dim \operatorname{Ran}(AB - BA) = 1$, we already know that all but one eigenvalues have to be zero. In order to compute the eigenvalues of $(AB - BA)$, we notice: Let λ be an eigenvalue of some matix $D \in M(n \times n, \mathbb{C})$. Then $D - \lambda \cdot E_n$ is not injective, therefore $\det(D - \lambda \cdot E_n) = 0$, which is an equation of a polynomial of n-th degree in the variable λ. Consequently, if we want to compute the eigenvalues of D, we have to find the roots of $\det(D - \lambda \cdot E_n) = 0$. This polynomial is called the *characteristic polynomial* $P_D(\lambda) = \det(D - \lambda \cdot E_n)$ of D.
Back to the problem: Set $C := (AB - BA) \in M(n \times n, \mathbb{C})$. We know, that $n-1$ eigenvalues are zero and one eigenvalue is different from zero. Thus the kernel of C has dimension $n-1$ and we can choose a base $W := \{w_1, \ldots, w_n\}$ of $\mathbb{C}^n$ with $C \cdot w_1 = \lambda \cdot w_1, C \cdot w_2 = \ldots = C \cdot w_n = 0$, set $S := (w_1 \ldots w_n)^{-1}$. Then

$$S \cdot C \cdot S^{-1} = \begin{pmatrix} \lambda & & & \\ & 0 & & \\ & & \cdots & \\ & & & 0 \end{pmatrix}$$

Now, we consider the *trace* $\operatorname{tr} A := \sum_{i=1}^{n} a_{ii}$ C. It is easy to show, that $\operatorname{tr} AB = \operatorname{tr} BA$ and $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$. We conclude

$$\operatorname{tr} C = \operatorname{tr} E_n C = \operatorname{tr} SCS^{-1} = \lambda = \operatorname{tr}(AB - BA) = \operatorname{tr} AB - \operatorname{tr} AB = 0$$

which contradicts our assumption.

For the last step, we used the concept of *normal forms*:
Given two vectorspaces V and W, both with finite dimensions, and a linear map $F : V \to W$, there exist bases B of V and C of W such that

$$M_C^B(F) = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$$

with $r := \dim \operatorname{Ran} F$. Consider the case $V = W$: Again, one could ask whether there exists a base B of V such that $M_B^B(F)$ is a "simple" representation, e.g. a diagonal form. In general, it is not possible to find such a base, but we can construct the *Jordan normal form.* Note, that we can induce an equivalence relation R on $M(n \times n, \mathbb{C})$:

$$(A, B) \in R :\Leftrightarrow \exists S \in M(n \times n, \mathbb{C}) : A = SBS^{-1}$$

If some matrix A is in the same equivalence class as B, B being a diagonal matrix, then A can also be transformed into the same diagonal form, therefore A has the same eigenvalues and the same rank as B. These information should be sufficient for solving the next problem:

Exercise 5:(IMC 1995)
Let X be a nonsingular matrix with columns $X_1, X_2, \ldots, X_n$. Let Y be a matrix with columns $X_2, X_3, \ldots, X_n, 0$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only $0's$ for eigenvalues.

1.4 Solutions of the exercises

Exercise 1: Barabara has a winning strategy: If Alan places some real number r in the $(i,j)^{th}$ entry, she simply places, given that i is even, the same number in the $(i-1,j)^{th}$ entry, otherwise in the $(i+1,j)^{th}$ entry.

Exercise 2: For $n = 1$ the rank is 1. Now assume $n \geq 2$. Since $A = (i)_{i,j=1}^{n} + (j)_{i,j=1}^{n}$, matrix A is the sum of two matrixes of rank 1. Therefore, the rank of A is at most 2. The determinant of the top-left 2×2 minor is -1, so the rank is exactly 2. Therefore, the rank of A is 1 for $n = 1$ and 2 for $n \geq 2$.

Exercise 3: The minimal rank is 2 and the maximal rank is n. To prove this, we have to show that the rank can be 2 and n but it cannot be 1.
(1) The rank is at least 2. Consider an arbitrary matrix $A = (a_{ij})_{i,j=1}^{n}$ with entries $1, 2, \ldots, n^2$ in some order. Since permuting rows or columns of a matrix does not change its rank, we can assume that $1 = a_{11} < a_{21} < \ldots < a_{n1}$ and $a_{11} < a_{12} < \ldots < a_{1n}$. Hence $a_{n1} \geq n$ and $a_{1n} \geq n$ and at least one of these inequalities is strict.
Then

$$\det \begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix} < 1 \cdot n^2 - n \cdot n = 0$$

so

$$\operatorname{rank} A \geq \operatorname{rank} \begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix}$$

(2) The rank can be 2. Let

$$T = \begin{pmatrix} 1 & 2 & \ldots & n \\ n+1 & n+2 & \ldots & 2n \\ \ldots & \ldots & \ldots & \ldots \\ n^2-n+1 & n^2-n+2 & \ldots & n^2 \end{pmatrix}$$

(3) The rank can be n, i.e. the matrix can be nonsingular. Put odd numbers into the diagonal, only even numbers above the diagonal and arrange the entries under the diagonal arbitrarily. Then the determinant of the matrix is odd, so the rank is complete.

Exercise 4: Let $C = AB - BA$. Since $\operatorname{rank} C = 1$, at most one eigenvalue of C is different from 0. Also $\operatorname{tr} C = 0$, so all the eigenvalues are zero. In the Jordan canonical form there can only be one 2×2 cage and thus $C^2 = 0$.

Exercise 5: Let $J = (a_{ij})$ be the $n \times n$ matrix where $a_{ij} = 1$ if $i = j+1$ and $a_{ij} = 0$ otherwise. The rank of J is $n-1$ and its only eigenvalues are $0's$. Moreover $Y = XJ$ and $A = YX^{-1} = XJX^{-1}$, $B = X^{-1}Y = J$. It follows that both A and B have rank $n-1$ with only $0's$ for eigenvalues.