Recursions

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1 The Method and the Lucas Numbers (Recursion Solving I)

Problem: we try to find a closed expression for some recurrence relation.

Constructing a method on an initial example: Recursion: $a_0 = 2$, $a_1 = 1$, $a_{n+1} = a_n + a_{n-1} \rightarrow$ called the Lucas-numbers after the french mathematician Edouard Lucas, who lived in the second half of the 19^{th} century.

1. Checking valid n's: valid $\forall n > 0$

2. Setting the generating Sequence:

$$L(x) = \sum_{n=1}^{\infty} a_n x^n$$

3. Now we differ from the known routine:

   - $a_{n+1} = a_n + a_{n-1}$ is valid for every n
   - $a_{n+1}x^n = anx^n + a_{n-1}x^n$ is trivially valid, too

   As it's valid for every n defined in (1), we can sum over these terms:

$$\sum_{n=1}^{\infty} a_{n+1}x^n = \sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} a_{n-1}x^n$$

4. Now we try to wrap this expression around until we can express it in terms of L(x). Finally:

$$L(x) = \frac{L(x) - a_0}{x} + \frac{L(x) - a_0 - a_1 x}{x^2}$$

5. Now we fill in our starting values and extract L(x); that is the generating function for this recurrence:

$$L(x) = \frac{2 - x}{1 - x - x^2}$$

6. To find the closed form we have to do a partial fraction expansion of this and express it as a power series. When we do this we get:

$$a_n = l_1^n + l_2^n \quad l_{1/2} = \frac{1 \pm \sqrt{5}}{2}$$

2 Periodicity and modulo

2.1 Problem 1

Let $a_1 = a_2 = 1$, $a_3 = -1$ and $a_n = a_{n-1}a_{n-3}$. What is a_{2011}?

If we calculate the first values of this sequence, we get:

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	-1	-1	-1	1	-1	1	1	-1

Because the computation of a_n only needs the last three terms of the series, the series is periodic with periodic length 7. With $2011 = 7 \cdot 287 + 2$ we get $a_{2011} = 1$.

Do the small cases. Look out for periodic patterns!

2.2 Problem 2

Let $a_1 = a_2 = 1$ and $a_n = a_{n-1}a_{n-2} + 1$. Show that $4 \nmid a_{2011}$.

Again we write down the first numbers of the sequence:

n	1	2	3	4	5	6	7	8
a_n	1	1	2	3	7	22	155	3411
$a_n \pmod 4$	1	1	2	3	3	2	3	3

Because we do not spot a pattern in the sequence for a_n, let's look at the sequence ($a_n \pmod 4$) (because we are interested in divisibility by 4). Now we easily see the the pattern $2, 3, 3$. Does this pattern continue? Because of

$$2 \cdot 3 + 1 = 7 \equiv 3 \pmod 4$$
$$3 \cdot 3 + 1 = 10 \equiv 2 \pmod 4$$
$$3 \cdot 2 + 1 = 7 \equiv 3 \pmod 4$$

we see that the pattern continues. Therefore the sequence ($a_n \pmod 4$) does not contain any zero, and no a_n is divisible by 4.

Focus on what is important for the problem!

3 Recursion Solving II

Incentive: We can now solve recursions of the form

$$a_n = a_{n-1} + a_{n-2} - 3n + 12...$$

or similar. New Problem: We want to solve

$$a_n = a_{n-1} + a_{n-2} \quad a_1 = 1a_{735} = 1$$

Naive attempt: Iterating down and hope to find a pattern.

$$a_{735} = a_{734} + a_{733} = a_{733} + 2a_{732} + a_{731} = ...$$

$\Rightarrow$ ineffective, we need a new tool!

First, we look at recursions of the form

$$a_n = pa_{n-1} + qa_{n-2}$$

We introduce a new term, the so called auxiliary polynomial or characteristic equation for a recurrence. $\rightarrow$ substitute a_n with λ^n

$$\lambda^n = p\lambda^{n-1} + q\lambda^{n-2}$$
$$\lambda^2 = p\lambda + q$$
$$\lambda^2 - p\lambda - q = 0$$

Does this look familiar? In fact it's the denominator of the generating function. We can now solve this as follows:
$a_n = r\lambda_1^n + s\lambda_2^n$ ($\lambda_{1/2}$ are the solutions to the upper equation)

For a little exercise, let's solve $a_0 = 2$, $a_1 = 7$, $a_n = 7a_{n-1} - 12a_{n-2}$ the characteristic equation is:

$$\lambda^2 - 7\lambda + 12 = 0 \quad \text{whereas} \quad \lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 4$$

so the general solution is $a_n = r3^n + s4^n$ but what are these r,s? As this equation (though we don't know r,s, yet) is generally true, we just can put in numbers as we want. So here we take n=0 and n=1.

$$a_0 = 2 = r + s$$
$$a_1 = 7 = 3r + 4s$$

$\Rightarrow r = s = 1$
Now, we're done, the closed form for the upper expression is: $a_n = 3^n + 4^n$

3.1 Exercises

1. $a_{n+1} = a_n + a_{n-1} \quad a_1 = 1$ and $a_{735} = 1$
2. $a_{n+1} + a_{n-2} = \sqrt{2}a_n$ Show that it's periodic and find the period.

3.2 Some useful relations

To give you a little tool set to work on recurrences, here are some relations

1. $a_{n+1} = ra_n$ has the general solution $a_n = a_1 r^{n-1}$
2. $a_{n+1} = a_n + s$ has the general solution $a_n = a_1 + (n-1)s$
3. $a_{n+1} = ra_n + s$, $(r \neq 1)$ can be formed into $a_{n+1} + k = r(a_n + k) \quad s = (r-1)k$ and so can be handled equally to (1)
4. $a_{n+1} = ra_n + sa_{n-1}$ has solutions depending on the discriminant of the characteristic function.
   (a) If $r^2 - 4s \neq 0$ and the distinct roots of the characteristic polynomial are l_1 and l_2, then the general solution of the recurrence is

   $$a_n = c_1 l_1^n + c^2 l_2^n$$

   (b) If $r^2 - 4s = 0$ and l is the double root of the characteristic polynomial, then we have the solution:

   $$a_n = (c_1^n + c_0)l^n$$

5. $a_{n+1} = (1-s)a_n + sa_{n-1} + r$ can be formed into $a_{n+1} - a_n = -s(a_n - a_{n-1}) + r$ and can be solved equally to (3) with $a_{n+1} - a_n$
6. $a_{n+1} = ra_n + sa_{n-1} + t$ where $(r + s \neq 1)$ can be rewritten $(a_{n+1} + k) = r(a_n + k) + s(a_{n-1} + k)$ where $(r + s - 1)k = t$ and solved like in (4) with $x_n + k$.

4 Partitions and Stirling Numbers

A partition of A is a collection of sets, so that these sets are nonempty, pairwise disjoint and their union is our set A.

The partitions of {1234} into two classes are (12)(34), (13)(24), (14)(23), (123)(4), (124)(3), (134)(2) and (234)(1).

How many partitions are there to divide n numbers into k classes?

Let $\left\{ {n \atop k} \right\}$ be the number of partitions of [n] into k classes. These numbers are called Stirling numbers of the second kind. Can we find a generating function or even an explicit formula for these numbers?

First of all we need a recurrence relation. We divide the partitions of [n] into k classes into two piles. In the first pile are all partitions where n has a class all by itself. In the second pile are all the recurrences, where n is not the only number in a class.

In the first pile are $\left\{ {n-1 \atop k-1} \right\}$ partitions. The number of partitions does not change, if we leave the class containing only n away. In the second pile are $k\left\{ {n-1 \atop k} \right\}$ partitions. If we remove n from all classes, the number of classes does not change, but we get each partition k times, because n was in everyone of these classes. From this we get our recurrence relation:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

We set $\left\{ {n \atop k} \right\} = 0$ if $k > n$, $k < 0$ or $n < 0$. $\left\{ {n \atop 0} \right\} = 0$ for all $n > 0$ and $\left\{ {0 \atop 0} \right\} = 1$. Now we have three possibilities to define our generating function.

$$A_n(y) = \sum_k \left\{ {n \atop k} \right\} y^k$$
$$B_k(x) = \sum_n \left\{ {n \atop k} \right\} x^n$$
$$C(x,y) = \sum_{k,n} \left\{ {n \atop k} \right\} x^n y^k$$

Let us take the generating function $B_k(x)$. We multiply the recurrence relation with x^n and sum on n. From the definition of $B_k(x)$ follows:

$$B_k(x) = xB_{k-1}(x) + kxB_k(x) \qquad (k \geq 1; B_0(x) = 1)$$

This leads to

$$B_k(x) = \frac{x}{1-kx} B_{k-1}(x)$$

and to our generating function

$$B_k(x) = \sum_n \left\{ {n \atop k} \right\} x^n = \frac{x^k}{(1-x)(1-2x)(1-3x)(1-4x) \cdot \ldots \cdot (1-kx)}$$

With the help of partial fractions we get an explicit formula for the Stirling numbers of the second kind.

$$\left\{ {n \atop k} \right\} = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!} \quad (n, k \geq 0)$$

If you have a recursion with more than variable, you have more than one generating function. Every variable of the recursions must be in the generating function either as variable or as parameter.

5 The Josephus problem and other Number Systems

5.1 The problem

Josephus, a famous historian, was trapped in a cave with 41 other man by the Romans. Because they preferred suicide to capture, they decided to form a circle and kill every second person until no one is left. Josephus wanted not kill himself so he calculated the position of the last remaining person.

We look at a more general case. Let n be the number of persons and $J(n)$ the number of the last survivor. Again we look at the small cases:

n	1	2	3	4	5	6
$J(n)$	1	1	3	1	3	5

$J(n)$ seems to be an odd number. This is because we killed all even persons on our first turn around the circle. Can we make a formula out of that? Let's say we had 2n person in the beginning after our first turn we killed the even persons and we have n persons left. The first person is skipped again. So we have exactly the same situation like if we had started with n persons. The only difference is that the number of the person is doubled and decreased by 1. This means:

$$J(2n) = 2J(n) - 1 \quad (n \geq 1)$$

If we had $2n+1$ persons in the beginning, person 1 is killed right after number $2n$. Now we have n persons in the circle. 3 is the person to be skipped next and we have again the situation as we had with two persons, but now the number of the person is doubled and increased by one. Therefore:

$$J(2n+1) = 2J(n) + 1 \quad (n \geq 1)$$

Sometimes a recursion can't be described through one equation.

With $J(1) = 1$ we can now calculate some more for our table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Out of this data we make a conjecture:

$$J(2^m + l) = 2l + 1 \quad (0 \leq l < 2^m)$$

This conjecture can be proved easily by induction (two cases! even, odd).

5.2 Other number Systems

There is no special reason why we took ten as a basis for our number system. So let's have a look at the problem in the number System with base 2. Let

$$n = (b_m b_{m-1} \dots b_1 b_0)_2$$

the binary expansion of n. Because 2^m is the largest power of 2 not exceeding n, we have $b_m = 1$.

$$\begin{aligned} n &= (b_m b_{m-1} \dots b_1 b_0)_2 \\ l &= (0 b_{m-1} \dots b_1 b_0)_2 \\ 2l &= (b_{m-1} \dots b_1 b_0 0)_2 \\ 2l + 1 &= (b_{m-1} \dots b_1 b_0 1)_2 \\ J(n) &= (b_{m-1} \dots b_1 b_0 b_m)_2 \end{aligned}$$

Therefore we have

$$J\left((b_m b_{m-1} \dots b_1 b_0)_2\right) = (b_{m-1} \dots b_1 b_0 b_m)_2$$

This means that all we have to do to calculate $J(n)$ is a cyclic one-bit left shift. $J(100) = J((1100100)_2) = (1001001)_2 = 73$

Sometimes a recursion can't be described through one equation.

6 Problems

Problems taken from Concrete Mathematics by Graham, Knuth and Patashnik and from generatingfunctionology by Wilf.

6.1 Towers of Hanoi

1. Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed. (Each move must be to or from the middle peg. As usual, a larger disk must never appear above a smaller one.)

2. A Double Tower of Hanoi contains $2n$ disks of n different sizes, two of each size. As usual we are required to move only one disk at a time, without putting a larger one over a smaller one.

   (a) How many moves does it take to transfer a double tower from one peg to another. If disks of equal size are indistinguishable from each other?

   (b) What if we are required to reproduce the original top-to-bottom order of all the equal-size disks in the final arrangement.

6.2 Stirling numbers and Binomial coefficients

1. The Binomial coefficient $\binom{n}{k}$ is the number of subsets (containing k elements) of a set containing n elements. Find a generating function and a explicit formula for the Binomial coefficients.

2. The Stirling numbers of the first kind $\left[{n \atop k}\right]$ are the numbers of permutations of n elements in k cycles. Find a generating function and a explicit formula for these numbers.

3. For given integers n, k, let $f(n, k)$ be the number of k-subsets of $[n]$ that contain no two consecutive elements. Find the recurrence that is satisfied by these numbers, find a suitable generating function and find the numbers themselves. Show the numerical values of $f(n, k)$ in a Pascal triangle arrangement, for $n \leq 6$.

6.3 Josephus and friends

1. In the original Josephus problem every third person was killed. Find a recurrence relation and a explicit formula. What do you get, if every k-th person is killed?

2. Josephus had a friend who was saved by getting into the next-to-last position. What is $I(n)$, the number of the penultimate survivor when every second person is executed?