Residue Calculus

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1 Classification of isolated singularities

Definition: Let $U \subseteq \mathbb{C}$ be open, $a \in U$ and $f \in U\backslash\{a\}$. We say f has an *isolated singularity* at a, which can be classified further:

1. f has a *removable singularity* at a, if there exists a holomorphic function $g : U \to \mathbb{C}$ which agrees with f on $U\backslash\{a\}$.
2. f has a *pole* at a , if $\lim\limits_{z \to a} |f(z)| \to \infty$.
3. f has an *essential singularity* at a, if f has neither a *removable singularity* nor a *pole* at a.

Examples of singularities:

1. The function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ with $f(z) = \frac{sin(z)}{z}$ has a *removable singularity* in 0, because we can set $g : \mathbb{C} \to \mathbb{C}$ with $g(z) = f(z) \quad \forall z \in \mathbb{C} \setminus \{0\}$ and $g(0) = 1$.
2. The function $f : \mathbb{C}\backslash\{-\frac{d}{c}\} \to \mathbb{C}$ with $f(z) = \frac{az+b}{cz+d}$ and $c \neq 0$ has a *pole* in $-\frac{d}{c}$, because $\lim\limits_{z \to \frac{-d}{c}} |f(z)| \to \infty$.
3. The function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ with $f(z) = e^{\frac{1}{z}}$ has an *essential singularity* in 0, because $\lim\limits_{z \to 0^+} |f(z)| \to +\infty$ and $\lim\limits_{z \to 0^-} |f(z)| \to 0$.

Theorem (Riemann‘s Theorem on Removable Singularities). Let $f \in H(U\backslash\{z_0\})$. If $f(z)$ is bounded in a punctured neighborhood of z_0, then f has a removable singularity at z_0.

Theorem (Casorati-Weierstrass). Let $z_0 \in U$ be an *essential singularity* of $f \in H(U\backslash\{z_0\})$, then $\forall w \in \mathbb{C}$ there exists a sequence $z_n \to z_0$ with $f(z_n) \to w$ and $n \to \infty$.

Lemma (Characterisation of a pole). Let U be an open environment of $z_0 \in \mathbb{C}$, $f \in H(U \setminus \{a\})$. The following statements are equivalent:
(i) a is a pole of f
(ii) There exists a unique number $n \in \mathbb{N}$ and a holomorphic function $g : U \to \mathbb{C}$, $g(a) \neq 0$ satisfying

$$f(z) = \frac{g(z)}{(z-a)^n}, \, \forall \, z \in U \setminus \{a\}$$

2 Laurent Series

A series of the form $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$, $a_k \in \mathbb{C} \quad \forall k \in \mathbb{Z}$ and $z, z_0 \in \mathbb{C}$ is called *Laurent Series.*

$$\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z-z_0)^k \; (*)$$

The first term is called *main-part* of a Laurent Series and the second one is called *secondary-part.* A Laurent Series is called *convergent (absolute convergent, uniformly convergent)*, if the main- and secondary-part are *convergent (absolute convergent, uniformly convergent).*

Remark

- $\frac{1}{r} \in [0, \infty]$ is the radius of convergence of $\sum_{k=1}^{\infty} a_{-k}(\beta)^k$, $\beta \in \mathbb{C}$
- $R \in [0, \infty]$ is the radius of convergence of $\sum_{k=0}^{\infty} a_k(\beta)^k$

$\Rightarrow$ the Laurent Series (*) is convergent on $K_{r,R}(z_0) := \{z \in \mathbb{C} | r < |z - z_0| < R\}$

Lemma. If the Laurent Series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

with $a_k \in \mathbb{C} \quad \forall k \in \mathbb{Z}$ and $z, z_0 \in \mathbb{C}$ is convergent on $K_{r,R}(z_0) := \{z \in \mathbb{C} | r < |z - z_0| < R\}$, then we can write for every $\gamma \in (r, R)$ and $n \in \mathbb{Z}$

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Proof. Without loss of generality let $z_0 = 0$.

$$\int_{|z|=\gamma} \frac{f(z)}{z^{n+1}} dz = \sum_{k=-\infty}^{\infty} a_k \int_{|z|=\gamma} z^{k-n-1} dz = 2\pi i \cdot a_n$$

The first equation results from the *uniform convergence* of a series inside of the *convergence radius* and therefore we can switch the sum and integral. The second equation follows from the integration of z^{k-n-1} on a circle. (see last week)

Lemma (Laurent Expansion). Let $f : K_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then we can write

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

with $a_k = \frac{1}{2\pi i} \int_{|z-z_0|=\gamma} \frac{f(z)}{(z-z)^{k+1}} dz \quad \forall k \in \mathbb{Z}$.

Proof. Without loss of generality let $z_0 = 0$. Using Cauchy's integral formula, we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\xi - z|=\varepsilon} \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{|\xi|=R-\delta} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{|\xi|=r+\delta} \frac{f(\xi)}{z - \xi} d\xi \\ &= \frac{1}{2\pi i} \int_{|\xi|=R-\delta} \frac{f(\xi)}{\xi} \cdot \frac{d\xi}{1 - \frac{z}{\xi}} + \frac{1}{2\pi i} \cdot \frac{i}{z} \int_{|\xi|=r+\delta} \frac{f(\xi)}{\xi} \cdot \frac{d\xi}{1 - \frac{\xi}{z}}; \delta > 0 \end{aligned}$$

Geometrical Series:

$$\begin{aligned} \frac{1}{1 - \frac{z}{\xi}} &= \sum_{k=0}^{\infty} (\frac{z}{\xi})^k \; |z| < R \\ \frac{1}{1 - \frac{\xi}{z}} &= \sum_{k=0}^{\infty} (\frac{\xi}{z} c)^k \; |z| > r \end{aligned}$$

Using the Geometrical Series and integrating component-by-component leads to the proposition.

Examples. $f(z) = \frac{1}{(z-1)(z-2)}$ is holomorphic on $\mathbb{C} \setminus \{\{1, 2\}$

- Laurent-Expansion in $K_{0,1}(0)$
  With expansion into partial fractions: $f(z) = \frac{1}{z-2} - \frac{1}{z-1}$

$$\begin{aligned} f(z) =& \frac{-1}{2(1 - \frac{z}{2})} + \frac{1}{(1 - z)} \\ =& \frac{-1}{2} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k + \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} (1 - 2^{-k-1}) z^k, \; \forall\, z \,\epsilon\, K_{0,1}(0) \end{aligned}$$

- Laurent-Expansion in $K_{1,2}(0)$

$$
\begin{aligned}
f(z) = \frac{1}{z-2} - \frac{1}{z-1} &= \frac{-1}{2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})} = \\
&= \frac{-1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k - \frac{1}{z}\sum_{k=0}^{\infty}(\frac{1}{z})^k = -\sum_{k=0}^{\infty} z^{-(k+1)} - \sum_{k=0}^{\infty} 2^{-(k+1)} z^k = \\
&= -\sum_{k=1}^{\infty} z^{-k} - \sum_{k=0}^{\infty} 2^{-(k+1)} z^k = \\
&= \sum_{k=-\infty}^{\infty} a_k z^k
\end{aligned}
$$

$$
with \quad a_k = \begin{cases} -1 & k < 0 \\ -2^{-(k+1)} & k \geq 0 \end{cases}
$$

- Laurent-Expansion in $K_{2,\infty}(0) = \{z \in \mathbb{C} : |z| > 2\}$.

$$
\begin{aligned}
f(z) = \frac{1}{z-2} - \frac{1}{z-1} &= \frac{1}{z(1-\frac{2}{z})} - \frac{1}{z(1-\frac{1}{z})} = \\
&= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k - \frac{1}{z}\sum_{k=0}^{\infty}(\frac{1}{z})^k = \sum_{k=0}^{\infty} 2^k z^{-(k+1)} - \sum_{k=0}^{\infty} z^{-(k+1)} = \\
&= \sum_{k=1}^{\infty} 2^{k-1} z^{-k} - \sum_{k=1}^{\infty} z^{-k} = \sum_{k=1}^{\infty} (2^{k-1} - 1)\, z^k \quad (|z| > 2)
\end{aligned}
$$

Lemma (Characterization of Singularities with Laurent Series). Let $f \in H(U\backslash\{a\})$. Then there exists a $R > 0$, with $K_{0,R}(a) \subset U\backslash\{a\}$ and $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ is convergent on $K_{0,R}(a)$. Furthermore:

1. The singularity a is removable, if and only if $a_k = 0$ for all $k \leqq -1$.(The main part of the Laurent Series is zero).
2. The singularity a is a pole of the order n, iff $a_{-n} \neq 0$ and $a_{-k} = 0$ for all $k > n$.
3. The singularity a is essential, iff $a_{-k} \neq 0$ for infinity in numbers $k \,\epsilon\, \mathbb{N}$.

3 Residue Calculus

Definition (Residue). Let $f : K_{0,r}(z_0) \rightarrow \mathbb{C}$ with $r > 0 \quad z_0 \in \mathbb{C}$ be holomorphic. For $\gamma \in (0, r)$ we define

$$Res_{z_0}(f) := \frac{1}{2\pi i} \int_{|z-z_0|=\gamma} f(z)dz = a_{-1}$$

the *residue* of f in z_0.

Properties of the Residue. Let the functions $h, f : K_{0,r}(z_0) \rightarrow \mathbb{C}$ with $r > 0, z_0 \in \mathbb{C}$ be holomorphic. Then

- $Res_{z_0}(f + h) = Res_{z_0}(f) + Res_{z_0}(h)$
- $Res_{z_0}(\lambda f) = \lambda Res_{z_0}(f)$ with $\lambda \in \mathbb{C}$

Calculation Rules:

1. If z_0 is a pole of order $n = 1$ of f, we can write: $Res_{z_0}(f) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$.
   Proof. We can write $f(z) = \frac{g(z)}{z-z_0}$ with $g(z_0) \neq 0$ and g holomorphic (Shown in 1.5). Using Taylor expansion of g in z_0, it follows that

   $$f(z) = \frac{1}{z - z_0}(g(z_0) + (z - z_0)g`(z_0) + ...) = \frac{g(z_0)}{z - z_0} + g`(z_0) + (z - z_0)[...]$$

   thus $Res_{z_0}(f) = g(z_0)$. But $g(z_0) = lim_{z \rightarrow z_0}(z - z_0)f(z)$.

2. If $f : U \backslash \{z_0\} \rightarrow \mathbb{C}$ has a pole in z_0 of the order m ($m \in \mathbb{N}$), so we can write $f(z) = \frac{g(z)}{(z-z_0)^m}$, $g : U \rightarrow \mathbb{C}$ holomorphic. Conclusively

   $$Res_{z_0}(f) = \frac{g^{(m-1)}(z_0)}{(m - 1)!}$$

   Proof.
   We can write $g(z) = a_0 + a_1(z - z_0) + ... + a_m(z - z_0)^m + ...$,as a power series of g in z_0, because every holomorphic function can be written as power series in $K_{r,R}(z_0)$. Therefore we can write $f(z) = \frac{a_0}{(z-z_0)^m} + ... + \frac{a_{m-1}}{z-z_0} + a_m + (z - z_0)[...]$ hence

   $$Res_{z_0}(f) = a_{m-1} = \frac{g^{(m-1)}(z_0)}{(m - 1)!}.$$

Theorem (Residue Calculus). Let U open in $\mathbb{C}$ and $S := \{z_1, z_2, ..., z_n\}$ a set of pairwise disjoint points $z_j \in U$. Let $f \in H(U \backslash S)$ and $\gamma : [0, 1] \rightarrow U \backslash S$ be closed, piecewise differentiable curve without overlaps, nullhomolog (it means that inner part of the curve is completely in U) in U and covered S with positive orientation (against the clockwise direction). Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res_{z_j}(f)$$

Proof. Let f_i be the main-part of the Laurent-Expansion from f of the point z_i $(i = 1, ..., n)$. We know that f_i is holomorphic in $\mathbb{C}\backslash\{z_i\}$. Let $f_i := \frac{a_i}{z-z_i} + f_i^*$, $a_i = Res_{z_i}(f)$; f_i^*, the Laurent-Series in which the coefficient a_{-1} disappears.$\Longrightarrow$ f_i^* has an antiderivative in $\mathbb{C}\backslash\{z_i\}$. Therefore we can write $\int_\gamma f_i^*(z)dz = 0$ and D

$$\int_\gamma f_i dz = a_i \int_\gamma \frac{1}{z - z_i} dz = 2\pi i \cdot a_i.$$

- Due to our construction, $f - (f_1 + f_2 + ... + f_n)$ is holomorphic in U and we have proposed that γ is nullhomolog, therefore we can use Cauchy‘s integral formula:

$$\int_\gamma [f - (f_1 + f_2 + ... + f_n)]dz = 0,$$

that is

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma f dz &= \sum_{i=1}^{n} \frac{1}{2\pi i} \int_\gamma f_i(z)dz = \\ &= \sum_{i=1}^{n} a_i = Res_{z_i}(f) \end{aligned}$$