Stochastic Programming

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Exercise sheet 4

Exercise 4.1, continued **Benders' decomposition:** Consider the **facility location problem**: A company wants to expand into new territories and considers m possible locations for new factories, the cost of opening a factory at location i is f_i. A market analysis has revealed the demand for each of the n potential customers, the cost of supplying customer j from factory i is c_{ij}. We want to determine at which locations a factory should be opened and which fraction of each customer's demand should be supplied from which factory in order to minimize the total cost (delivery plus factory setup).

Use variables $x_{ij} \in [0;1]$ to determine which fraction of customer j's demand should be supplied from the factory at location i and variables $y_i \in \{0,1\}$ to determine whether or not to open a factory at location i.

1. Design an IP model for the facility location using the notation given above.
2. We want to apply Benders' decomposition to the problem. Formulate the dual subproblem for fixed y-variables. How can it be solved? Outline how to find a feasibility cut and an optimality cut.

Solution

1. One possible IP model is:

$$
\begin{aligned}
\min\ & \sum_{i,j} c_{ij}x_{ij} + \sum_i f_i y_i \\
\text{s.t. } & x_{ij} \leq y_i,\ \forall i,j \\
& \sum_i x_{ij} \geq 1,\ \forall j \\
& x \geq 0 \\
& y \in \{0,1\}^m
\end{aligned}
$$

   Note that $x_{ij} \leq 1$ is implied by $x_{ij} \leq y_i$ and $y \in \{0,1\}^m$, so we do not need to write down that constraint explicitly. Also note that as an alternative to the constraints $x_{ij} \leq y_i$, one may use constraints of the type $\sum_j x_{ij} \leq y_i$. However, the constraints used in the model above will have an advantage in that the dual subproblem becomes a little easier to solve, so we decided in favor of the above formulation.

2. We rewrite the IP slightly. First define $q(y)$ as

$$
\begin{aligned}
q(y) := \min\ & \sum_{i,j} c_{ij} x_{ij} && (1)\\
\text{s.t. } & -x_{ij} \geq -y_i,\ \forall i,j && (2)\\
& \sum_{i} x_{ij} \geq 1,\ \forall j && (3)\\
& x \geq 0,
\end{aligned}
$$

then the IP can be written as

$$
\begin{aligned}
\min\ & \sum_i f_i y_i + q(y)\\
\text{s.t. } & y \in \{0,1\}^m.
\end{aligned}
$$

(note, that y is the decision on the locations, which might be set before chosing the percentage of demand). The latter problem will later serve as our first relaxed master, the dual of the former problem will constitute our dual subproblem. To dualize, we attach dual variables w_{ij} to the constraints (2) and dual variables v_j to the constraints (3). For a fixed y-value of $\overline{y}$ we then get the dual subproblem

$$
\begin{aligned}
q(\overline{y}) = \max\ & \sum_{i,j} -\overline{y}_i w_{ij} + \sum_j v_j\\
\text{s.t. } & -w_{ij} + v_j \leq c_{ij},\ \forall i,j\\
& v, w \geq 0.
\end{aligned}
$$

For those values of i where $\overline{y}_i = 0$, the w_{ij} do not change the objective value at all, so we are free to set these to any suitable nonnegative value, while for $\overline{y}_i \neq 0$ we might want to set the corresponding w_{ij} to 0. This quickly leads to the following proposition:

Proposition An optimal solution (w^*, v^*) to the dual subproblem is given by

$$
\begin{aligned}
v_j^* &:= \min\{c_{ij} : \overline{y}_i = 1\},\\
w_{ij}^* &:= \begin{cases} 0, & \text{if } \overline{y}_i = 1,\\ \max\left\{v_j^* - c_{ij}, 0\right\}, & \text{if } \overline{y}_i = 0. \end{cases}
\end{aligned}
$$

A corresponding primal optimal solution x^* can simply be found by assigning each customer to the factory closest (with respect to c - minimize delivery costs!) to his/her location.

Note that the problem has a solution (i.e. is not unbounded) whenever $\overline{y} \neq 0$. As we start with a non-zero value for $\overline{y}$ in this example, there will never be a Benders' feasibility cut. For the optimality cuts, see the example in the next part.

3. Benders' decomposition basically does the following:

   a) Solve the relaxed master and fix $\overline{y}$. The relaxed master is a relaxed version of the original problem and hence always yields a lower bound for the objective value of the optimum.

b) For fixed $\overline{y}$, solve the dual subproblem. Its objective value plus $\sum_i f_i y_i$ gives an upper bound for the objective value of the optimum (of course, this will be reversed when we start with a maximization problem instead of a minimization problem).

c) If the two bounds coincide, we are done. Otherwise, the dual optimal solution constitutes a (lower) bound that is valid for *all* y-values (because y does only appear in the dual objective), which yields a Benders' optimality cut that is added to the relaxed master problem.

d) The process is repeated until it yields an optimal solution.

4. Example:

factory location	delivery cost to customers					fixed setup cost
	1	2	3	4	5	
1	2	3	4	5	7	2
2	4	3	1	2	6	3
2	5	4	2	1	3	3.

Start with $y = (1, 0, 0)$.

In our example, we start with the relaxed master

$$\begin{aligned} \min \ & \sum_i f_i y_i + q(y) \\ \text{s.t. } & y \in \{0,1\}^m \end{aligned}$$

and an optimal solution $\overline{y} = (1, 0, 0)^T$, $q^* = -\infty$. Plugging this into the dual subproblem yields

$$\begin{aligned} q(\overline{y}) = \max \ & \sum_j -w_{1j} + \sum_j v_j \\ & - w_{ij} + v_j \le c_{ij} \quad \text{for all } i, j \\ & v, w \ge 0 \end{aligned}$$

for which an optimal solution is given by

$$\begin{aligned} v^* &= (2, 3, 4, 5, 7)^T, \\ w_1^* &= (0, 0, 0, 0, 0)^T, \\ w_2^* &= (0, 0, 3, 3, 1)^T, \\ w_3^* &= (0, 0, 2, 4, 4)^T. \end{aligned}$$

The dual objective value is $q(\overline{y}) = 21$, thus $q(\overline{y}) + \sum_i f_i \overline{y}_i = 21 + 2 = 23 \neq \sum_i f_i \overline{y}_i + q^* = -\infty$. The bounds do not coincide, so we have to add an optimality cut that states that q has to be at least as large as the dual objective for arbitrary y (but our optimal w^*, v^*). Thus plugging w^*, v^* into the dual objective, we get

$$q \ge -\sum_j w_{2j}^* y_2 - \sum_j w_{3j}^* y_3 + \sum_j v_j^* = -7y_2 - 10y_3 + 21$$

The new relaxed master problem is now

$$\begin{aligned} \min \sum_i f_i y_i &+ q(y) \\ q &\geq -7y_2 - 10y_3 + 21 \\ y &\in \{0,1\}^m \end{aligned}$$

The optimal solution to this is $\overline{y} = (0,1,1)^T$, $q^* = 4$, yielding an objective value of $6+4=10$. The optimal solution to the dual subproblem for this fixed value of $\overline{y}$ is

$$\begin{aligned} v^* &= (4,3,1,1,3)^T, \\ w_1^* &= (2,0,0,0,0)^T, \\ w_2^* &= (0,0,0,0,0)^T, \\ w_3^* &= (0,0,0,0,0)^T. \end{aligned}$$

This yields $\sum_i f_i \overline{y}_i + q(\overline{y}) = 3+3+12 = 18 \neq 10$. Again, the solution is not optimal. (Remark: Note that, although we do not yet have an optimal solution, we do have a feasible solution and we even an estimate of how much it could be improved in the best case!) The values of w^* and v^* provide for another optimality cut:

$$q \geq -2y_1 + 12$$

The next relaxed master problem is therefore

$$\begin{aligned} \min \sum_i f_i y_i &+ q(y) \\ q &\geq -7y_2 - 10y_3 + 21 \\ q &\geq -2y_1 + 12 \\ y &\in \{0,1\}^m, \end{aligned}$$

an optimal solution is $\overline{y} = (0,0,1)^T$, $q^* = 12$ with objective value $3+12=15$. The optimal solution to the dual subproblem for this fixed value of $\overline{y}$ is

$$\begin{aligned} v^* &= (5,4,2,1,3)^T, \\ w_1^* &= (3,1,0,0,0)^T, \\ w_2^* &= (1,1,1,0,0)^T, \\ w_3^* &= (0,0,0,0,0)^T. \end{aligned}$$

This yields $\sum_i f_i \overline{y}_i + q(\overline{y}) = 3+15 = 18 \neq 15$, still no optimal solution. We thus get another optimality cut:

$$q \geq -4y_1 - 3y_2 + 15$$

The yields a new relaxed master problem:

$$\begin{aligned}\min \ & \sum_i f_i y_i + q(y)\\ & q \geq -7y_2 - 10y_3 + 21\\ & q \geq -2y_1 + 12\\ & q \geq -4y_1 - 3y_2 + 15\\ & y \in \{0,1\}^m\end{aligned}$$

Computing an optimal solution yields $\overline{y} = (1,0,1)^T$, $q^* = 11$, the objective value is $2+3+11 = 16$. The optimal solution to the dual subproblem for this fixed value of $\overline{y}$ is

$$\begin{aligned}v^* &= (2,3,2,1,3)^T,\\ w_1^* &= (0,0,0,0,0)^T,\\ w_2^* &= (0,0,1,0,0)^T,\\ w_3^* &= (0,0,0,0,0)^T.\end{aligned}$$

We thus get $\sum_i f_i \overline{y}_i + q(\overline{y}) = 2 + 3 + 11 = 16 = 16$. Finally, both bounds coincide, we have therefore found an optimal solution for the y variables, namely $y^* = (1,0,1)^T$. Computing the corresponding x^*-values is now just a matter of solving an LP, we get

$$\begin{aligned}x_1^* &= (1,1,0,0,0)^T\\ x_3^* &= (0,0,0,0,0)^T\\ x_3^* &= (0,0,1,1,1)^T\end{aligned}$$

And, just to be sure, let us compute the total cost of facility location and delivery, which yields $c_{11} + c_{12} + c_{33} + c_{34} + c_{35} + f_1 + f_3 = 2+3+2+1+3+2+3 = 16$. Luckily, that value is the same as our upper and lower bound.

Exercise 4.2 **The L-Shaped Method**: Consider Step 3 of Iteration 1 within Example 1.

a) Let $\xi = \xi_1$, $x = x^1$ and

$$\begin{aligned}w = \min \ & -24y_1 - 28y_2\\ & 6y_1 + 10y_2 \leq 2400\\ & 8y_1 + 5y_2 \leq 1600\\ & 0 \leq y_1 \leq 500\\ & 0 \leq y_2 \leq 100.\end{aligned}$$

i) Check that the optimal dictionary (the simplex tableau in the optimal solution) is

$$\begin{aligned} w &= -6100 + 3s_2 + 13s_4 \\ s_1 &= 575 + 6/8s_2 + 50/8s_4 \\ y_1 &= 137.5 - 1/8s_2 + 5/8s_4 \\ s_3 &= 362.5 + 1/8s_2 - 5/8s_4 \\ y_2 &= 100 - s_4 \end{aligned}$$

for slack variables s_1, s_2, s_3, s_4.

ii) Check that this dictionary corresponds to the solution stated in Example 1.

iii) Check that the optimal value $w = -6100$ is also atained through the dual variables.

b) For $\xi = \xi_1$, the optimal solution is $w_1 = -6100$ and for $\xi = \xi_2$, the optimal solution is $w_2 = -8384$.

i) Check that $w^1 = 0.4w_1 + 0.6w_2$.

ii) Prove by linear programming duality that $w = \sum_{k=1}^{K} p_k w_k$, where w_k denotes the solution of the second-stage program, for realization k of ξ, $k = 1, \ldots, K$.

Solution

a) i) Let the problem $\max c_B A_B + c_N A_N$, $A_B x_B + A_N x_N = b$, $x \geq 0$ be given, then after one Pivot-step:

		$c_N^\intercal - c_B^\intercal A_B^{-1} A_N$	$-c_B^\intercal A_B^{-1} b$
x_B	I	$A_B^{-1} A_N$	$A_B^{-1} b$

.

	s_1	s_2	s_3	s_4	y_1	y_2	b
					-24	-28	
s_1	1				6	10	2400
s_2		1			8	5	1600
s_3			1		1		500
s_4				1		1	100
	s_1	s_2	s_3	y_2	y_1	s_4	b
					-24	28	2800
s_1	1				6	-10	1400
s_2		1			8	-5	1100
s_3			1		1		500
y_2				1		1	100
	s_1	y_1	s_3	y_2	s_2	s_4	b
					3	13	6100
s_1	1				$-6/8$	$-25/4$	575
y_1		1			$1/8$	$-5/8$	137 1/2
s_3			1		$-1/8$	$5/8$	362 1/2
y_2				1		1	100

ii) The optimal value w and the optimal solution x_B^* are given by:

$$\begin{aligned} w = & \; c_B^\intercal(A_B^{-1}b - A_B^{-1}A_N x_N) + c_N^\intercal x_N \\ = & \; c_B A_B^{-1} b + (c_N \intercal - c_B^\intercal A_B^{-1} A_N) x_N \\ = & \; -6100 \\ x_B^* = & \; A_B^{-1} b - A_B^{-1} A_N x_N \\ = & \; (137\,1/2, 100)^\intercal. \end{aligned}$$

The optimal dual solution can be computed via $w = c_B^\intercal x_B = (c_B^\intercal A_B^{-1})b$ where $\pi^* := c_B^\intercal A_B^{-1} = (0, -3, 0, -13)$.

iii) $\pi^* b = (0, -3, 0, -13)(2400, 1600, 500, 100)^\intercal = -6100$.

b) i)

$$w^v = e_1 - E_1 x^v = \sum_{k=1}^{K} p_k (\pi_k^v)^\intercal h_k - \left(\sum_{k=1}^{K} p_k (\pi_k^c)^\intercal T_k \right) x^v$$

Together with $x^1 = (40, 20)^\intercal$, $\pi_1^1 = \begin{pmatrix} 0 \\ -3 \\ 0 \\ -13 \end{pmatrix}$, $\pi_2^1 = \begin{pmatrix} -2.32 \\ -1.76 \\ 0 \\ 0 \end{pmatrix}$, $p_1 = 0.4$, $p_2 = 0.6$, $h_1 = \begin{pmatrix} 0 \\ 0 \\ 500 \\ 100 \end{pmatrix}$, $h_2 = \begin{pmatrix} 0 \\ 0 \\ 300 \\ 300 \end{pmatrix}$ and $T_1 = T_2 = \begin{pmatrix} -60 & 0 \\ 0 & -80 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ we have

$w^1 = 0.4(-1300 \; - 4800) + 0.6(-8384) = -6100 \cdot 0.4 - 8384 \cdot 0.6 = w_1 \cdot p_1 + w_2 \cdot p_2$.

ii) By duality we get

$$\begin{aligned} w^v = & \; e_{s+1} - E_{s+1} x^v = \sum_{k=1}^{K} p_k (\pi_k^v) \intercal h_k - (\sum_{k=1}^{K} p_k (\pi_k^v) \intercal T_k) x^v \\ = & \; \sum_{k=1}^{K} p_k \left((\pi_k^v)^\intercal h_k - (\pi_k^v)^\intercal T_k x^v \right) \\ = & \; \sum_{k=1}^{K} p_k (c_B^{v,\intercal} A_B^{v,-1})^\intercal \left(b_k^v + T_k x^v - T_k x^v \right) \\ = & \; \sum_{k=1}^{K} p_k (c_B^{v,\intercal} A_B^{v,-1})^\intercal b_k^v = \sum_{k=1}^{K} p_k w_k \end{aligned}$$

with the second-stage definition $Wy = (h_k - T_k x^v) =: b_k$ used in the Simplex method.

Exercise 4.3 Consider the following problem:

$$
\begin{aligned}
w = \min\ & 7x_1 + 11x_2 + E_\xi(q_1 y_1 + q_2 y_2) \\
& y_1 + 2y_2 \geq d_1 - x_1 \\
& y_1 \geq d_2 - x_2 \\
& 0 \leq x_1 \leq 10 \\
& 0 \leq x_2 \leq 10 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

where $\xi^\intercal = (q_1, q_2, d_1, d_2)$ takes on the values $(26, 16, 6, 12)$ and $(14, 24, 10, 4)$ with probability 0.5 each.

In this example, the L-Shaped method selects $x^1 = (0, 0)^\intercal$ as starting point. The L-Shaped method can however be used with any other reasonable starting point. Take $x^1 = (1, 5)^\intercal$ as starting point and show that exactly the same steps are taken if the starting point is any point within the region $4 \leq x_2 \leq 6 + x_1$.

Solution We start with Iteration 1 of the L-Shaped method. In step 1 we select $x^1 = (1, 5)^\intercal$ and observe the secon-stage to be solveable for all solutions of $x_1 \geq 0$ and $x_2 \geq 0$, because there is no upper bound on the values of y_1 and y_2.

We get for $k = 1$

$$
\begin{aligned}
\min\ & 26y_1 + 16y_2 \\
& y_1 + 2y_2 \geq 6 - 1 = 5 \\
& y_1 \geq 12 - 5 = 7 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

Every solution of the second-stage problem must satisfy the second inequality independent from the objective value of y_1. Therefore if the lower bound of y_1, i.e. $y_1 = 12 - x_2$, already satisfies the first inequality, then y_2 is set to 0. This happens if and only if $12 - x_2 \geq 6 - x_1$ which is equivalent with $x_2 \leq 6 + x_1$.

For $k = 2$ we observe

$$
\begin{aligned}
\min\ & 14y_1 + 24y_2 \\
& y_1 + 2y_2 \geq 10 - 1 = 9 \\
& y_1 \geq 4 - 5 = -1 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

Setting y_1 to a non-zero value is relatively cheaper than setting y_2 to a non-zero value. But the coefficient of y_2 in the first inequality is larger than y_1 (even relatively to the objective value). Hence y_1 is only choosen to be non-zero is there is a lower bound (bigger than 0) according to the second inequality. This does not happen if and only if $4 - x_2 \leq 0$ which is equivalent with $x_2 \geq 4$.