Discrete Optimization (MA 3502)

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Exercise Sheet 1

Exercise 1.1 (Rounding Solutions)
Let $a \in \mathbb{Z}^2$, $\beta \in \mathbb{Z}$ and consider the polyhedron

$$P(a,\beta) := \left\{x \in \mathbb{R}^2 : a^T x \le \beta, x \ge 0\right\}$$

and its *integer hull* $I(P)$ defined as

$$I(P) := \operatorname{conv}(P \cap \mathbb{Z}^2).$$

We want to solve the integer linear program $\max_{x \in P(a,\beta)} c^T x$ for some integral objective vector $c \in \mathbb{Z}^2$. Let x^* denote an optimal integer solution to this ILP and let $x' \in \mathbb{R}^2$ be an optimal solution of the LP relaxation, i. e. the ILP without the integrality constraints. In this exercise, we investigate the strategy of simply rounding x' down componentwise (denoted by $\lfloor x' \rfloor$) to get an integer solution.

a) Is $x' = x^*$ possible? Either give an example or disprove the statement!

b) Is it possible that $x' \neq x^*$, but $\lfloor x' \rfloor = x^*$? Either give an example or disprove the statement!

c) Show that $\lfloor x' \rfloor \in I(P(a,\beta))$ holds if $a, \beta \ge 0$.

d) Give an example that shows that x^* and $\lfloor x' \rfloor$ need not be "close" (both with respect to Euclidean distance and with respect to the objective value).

Answer to Exercise 1.1

a) Take $c = (1,0)^T$, $a = (1,1)^T$ and $\beta = 1$. Then, $x' = (1,0)^T = x^*$, thus the relaxation readily yields an integral solution.

b) Take $c = (1,0)^T$, $a = (2,3)^T$ and $\beta = 3$. Then the relaxed solution is $x' = (1.5,0)^T$ and rounding yields $\lfloor x' \rfloor = (1,0)^T$, which is also the integral solution.

c) Let x' be any feasible solution to the LP relaxation. Then

$$a^T \lfloor x' \rfloor \le a^T x' \le \beta,$$

thus $\lfloor x' \rfloor$ is still a feasible solution to the LP. As it is also integral, it is contained in $I(P(a,\beta))$.

d) Take $c = (21,11)^T$, $a = (7,4)^T$ and $\beta = 13$. Then, $x' = (\frac{13}{7},0)^T$, $\lfloor x' \rfloor = (1,0)^T$, but $x^* = (0,3)^T$. With $c^T x^* = 33$ and $c^T \lfloor x' \rfloor = 21$, those solutions are not very close to each other.

Exercise 1.2 (Simplex Tableau Revisited)
Consider the following linear program:

$$\begin{aligned} \max\ & 5x_1 + 6x_2 \\ & 3x_1 + 5x_2 \le 15 \\ & 3x_1 - 5x_2 \le 0 \end{aligned}$$

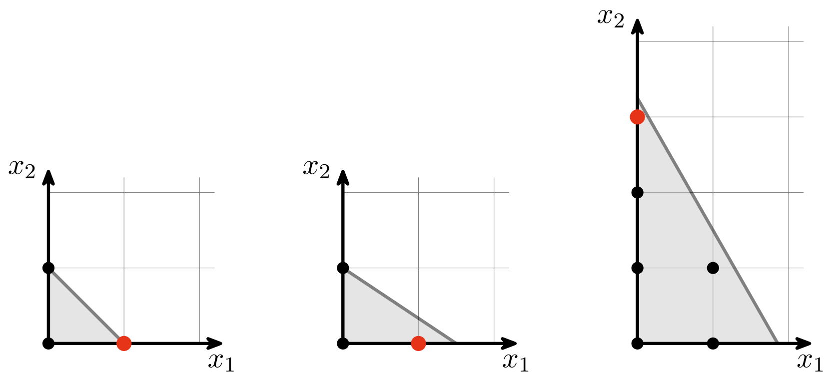


Figure 1: Illustration of the examples.

a) Sketch the feasible region for the above LP and guess an optimal solution.

b) Write down the dual of the above LP.

c) Prove optimality of your primal solution by devising a corresponding dual optimal solution.

d) Show how to compute the solution using the dual simplex method in tableau form.

e) The linear program is now modified by appending the inequality $2x_1 + x_2 \leq 6$. Show that this modification invalidates your current primal solution. Use the dual simplex method in tableau form to compute a new dual and primal solution. Can you re-use some of the work you did in the previous part of this exercise?

Answer to Exercise 1.2

a) The sketch is depicted in Figure 2

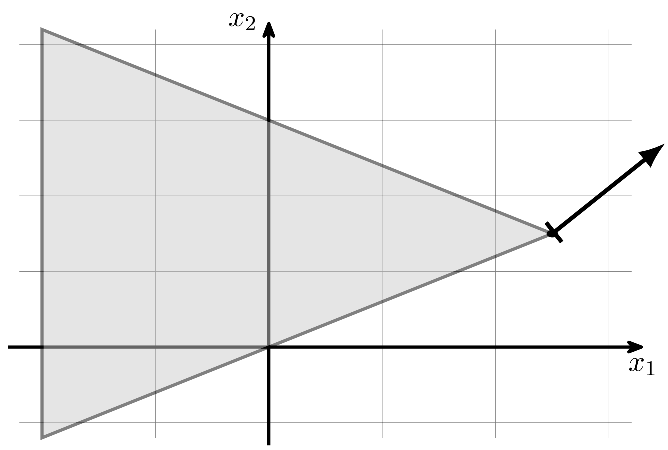


Figure 2: Feasible region of the LP.

The point $x^* = (2.5, 1.5)$ is feasible and seems to be the optimal solution for the LP.

b) The dual linear program is

$$\begin{aligned} &\min\ 15y_1 \\ &\quad 3y_1 + 3y_2 = 5 \\ &\quad 5y_1 - 5y_2 = 6 \\ &\quad y \geq 0 \end{aligned}$$

c) The dual equation system can easily be solved using Gaussian elimination:

$$\left(\begin{array}{cc|c} 3 & 3 & 5 \\ 5 & -5 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & 3 & 5 \\ 0 & -10 & -\frac{7}{3} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{43}{30} \\ 0 & 1 & \frac{7}{30} \end{array}\right)$$

This yields $y^* = \frac{1}{30}(43, 7)^T \geq 0$, so the solution is also feasible and hence optimal. Primal optimality of x^* is then easily proved by computing the primal and dual objective value which is $c^T x^* = 5 \cdot 2.5 + 6 \cdot 1.5 = 21.5 = \frac{15}{30} \cdot 43 = b^T y^*$.

d) While in this case we know both a dual and a primal solution, we nevertheless start the simplex tableau in phase I just to remind ourselves how the simplex tableau actually works. The simplex tableau for the problem min $c^T y$ subject to $Ay = b$, $y \geq 0$ in general looks like this:

$$\begin{array}{c|c} -c^T y & c - c_B^T A_B^{-1} A \\ \hline A_B^{-1} b & A_B^{-1} A \end{array}$$

For phase I of the simplex algorithm, we add one artificial variable to each of the two constraints so that we can use these artificial variables as a starting basis. We also use a different objective function for that first phase that aims at removing the artificial variables from the basis (thus making them take a value of 0) so that we can then remove those variables and start another round of the simplex algorithm on the remaining problem (phase II). Again, this step can easily be skipped using the information we have at this point (and as we already know the solution, the whole simplex could actually be skipped), but this exercise is about giving you a short recap of the simplex algorithm, so we will nevertheless do it. The auxiliary phase I problem looks like this after introduction of the two artificial variables y_3, y_4:

$$\begin{aligned} &\min\ y_3 + y_4 \\ &\quad 3y_1 + 3y_2 + y_3 = 5 \\ &\quad 5_1 - 5y_2 + y_4 = 6 \\ &\quad y \geq 0 \end{aligned}$$

It is easy to see that $y = (0, 0, 5, 6)^T$ is a vertex corresponding to the basis $B = \{3, 4\}$. The corresponding simplex tableau then looks like this (note that we add an extra row for the original objective function as we will need that later in phase II):

	y_1	y_2	$\mathbf{y_3}$	$\mathbf{y_4}$
0	15	0	0	0
0	0	0	1	1
5	3	3	1	0
6	5	−5	0	1

The first step is to compute the correct values for the reduced costs and the objective value (this last one can serve as a means of easily checking your computations at that point). To do this, we simply use elementary matrix operations to produce a 0-entry in the reduced cost vector corresponding to the two basis columns. In this case, we substract rows 1 and 2 from the cost row to get the reduced cost row:

	y_1	y_2	$\mathbf{y_3}$	$\mathbf{y_4}$
0	15	0	0	0
-11	-8	2	0	0
5	3	3	1	0
6	5	-5	0	1

Next, we check for columns with negative reduced cost. In this case, only column 1 has that, so this will be our pivot column, i. e., the next basis exchange will bring column 1 into the basis. To see which element will have to leave the basis, we determine the row i that takes the value

$$\min_{i:(A_B^{-1}A)_{i,1}>0} \frac{(A_B^{-1}b)_i}{(A_B^{-1}A)_{i,1}}$$

where 1 is our pivot column. (Note that if there are no positive values here then the problem will be unbounded.) In the tableau, we just divide each positive entry in the pivot column by the right hand side (which is in the leftmost column of the simplex tableau). The row with the minimal quotient will be our pivot element (and the corresponding basis element will have to leave the basis). In our tableau, we get

$$\begin{aligned} &\frac{5}{3} \quad \text{for row 1} \\ \text{and} \quad &\frac{6}{5} \quad \text{for row 2,} \end{aligned}$$

thus our pivot element ist $(2, 1)$ and y_4 will leave the basis. To actually perform the basis exchange, we use Gaussian elimination to change column 1 into a unit vector with 1-entry at position $(2, 1)$ (basically, we do the matrix inversion in this step). Then, we use elementary row operations to obtain a reduced cost value of 0 to compute the new reduced cost vector. As we will need the reduced cost vector corresponding to our original objective function in a later step, we do the same computation on the reduced cost row as well. This yields the following new simplex tableau:

	$\mathbf{y_1}$	y_2	$\mathbf{y_3}$	y_4
-18	0	15	0	-3
$-7/5$	0	-6	0	$8/5$
$7/5$	0	6	1	$-3/5$
$6/5$	1	-1	0	$1/5$

We now get a negative reduced cost value in column 2, which is quite lucky as we intended to get somehow get to basis $\{1, 2\}$ from the very start. So let us perform another iteration of the simplex algorithm to pivot y_2 into the basis. In this case, only position $(1, 2)$ has a positive entry, so we need not consider any other candidates for the pivot element. We also see that this pivot step will remove y_3 from the basis which should take us to the end of phase I. The result of this simplex step is the following tableau:

	$\mathbf{y_1}$	$\mathbf{y_2}$	y_3	y_4
$-43/2$	0	0	$-15/6$	$-3/2$
0	0	0	1	1
$7/30$	0	1	$1/6$	$-3/30$
$43/30$	1	0	$1/6$	$3/30$

The objective value of 0 for the auxiliary objective function tells us that phase I was actually successful, so we can now drop the artificial variables together with the auxiliary objective to arrive at a starting tableau for the simplex algorithm, phase II:

	$\mathbf{y_1}$	$\mathbf{y_2}$
$-43/2$	0	0
$7/30$	0	1
$43/30$	1	0

As there is no negative reduced cost value we already have an optimal solution, more precisely $y = (43/30, 7/30)$ with an objective value of $43/2$. Not surprisingly, this is the same solution that we had already obtained in an earlier step.

e) Adding the inequality $2x_1 + x_2 \leq 6$ to the primal problem yields the following modified dual:

$$\begin{aligned} \min\ & 15y_1 + 6y_3 \\ & 3y_1 + 3y_2 + 2y_3 = 5 \\ & 5y_1 - 5y_2 + y_3 = 6 \\ & y \geq 0 \end{aligned}$$

The primal solution $x^* = (5/2, 3/2)$ becomes infeasible, as $2x_1^* + x_2^* = 5 + 3/2 = 13/2 > 6$, but a feasible dual solution is easily obtained by simply setting $y_3 = 0$, i. e. $y = (43/30, 7/30, 0)$. Using this and the starting basis $\{1, 2\}$ we can write down the new simplex tableau right away by just appending a third column and then doing the Gaussian elimination steps to get to the form for basis columns 1 and 2:

	$\mathbf{y_1}$	$\mathbf{y_2}$	y_3
0	15	0	6
5	3	3	2
6	5	-5	1

$\rightsquigarrow$

	$\mathbf{y_1}$	$\mathbf{y_2}$	y_3
-18	0	15	3
$7/5$	0	6	$7/5$
$6/5$	1	-1	$1/5$

$\rightsquigarrow$

	$\mathbf{y_1}$	$\mathbf{y_2}$	y_3
$-43/2$	0	0	$-1/2$
$7/30$	0	1	$7/30$
$43/30$	1	0	$13/30$

As the third column has a negative reduced cost term, we can pivot y_3 into the basis. The pivot element is again determined by comparing the quotients, in this case it yields position $(1, 3)$ as the pivot element. Computing this simplex step yields the new tableau

	$\mathbf{y_1}$	y_2	$\mathbf{y_3}$
-21	0	$15/7$	0
1	0	$30/7$	1
1	1	$-13/7$	$13/30$

This yields the new dual optimal solution $y^* = (1, 0, 1)^T$. To obtain a corresponding primal solution we use complementary slackness. With y_1 and y_3 being the dual basis variables, the corresponding primal constraints will have to hold at equality, hence we know the primal optimum x^* will need to satisfy

$$3x_1 + 5x_2 = 15$$
$$2x_1 + x_2 = 6,$$

a quick computation yields $x^* = (15/7, 12/7)$.

Exercise 1.3 (The Matching Polytope)
Let $G = (V, E)$ be a graph on n vertices and m edges. Let S_G denote the node-edge incidence matrix of G and let $\mathcal{M}(G)$ denote the *matching polytope* of G, i. e. the convex hull of all feasible matchings of G:

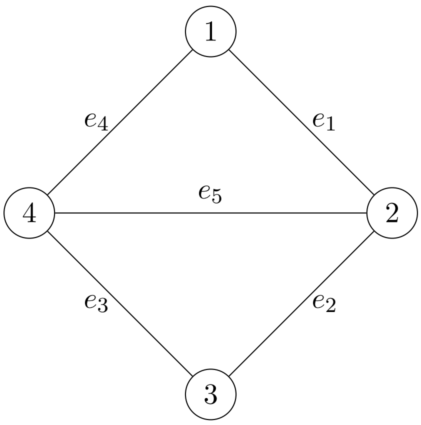
$$\mathcal{M}(G) := \operatorname{conv}(\{x \in \{0,1\}^m : S_G x \leq \mathbb{1}\})$$

Further, let $P = \{x \in \mathbb{R}^m : S_G x \leq \mathbb{1}, x \geq 0\}$ denote the LP relaxation of $\mathcal{M}(G)$.

a) Determine the dimension $\dim(\mathcal{M}(G))$.

b) Show that all inequalities of the form $x_e \geq 0$ define a facet of $\mathcal{M}(G)$.

In the following, consider the example of the graph depicted below.



c) Determine the outer normal cone of P at the vertex $x^* = (1/2, 0, 0, 1/2, 1/2)^T$.

d) Consider the vertex $x' := (0, 1/2, 1/2, 0, 1/2)$. Show that the inequality

$$x_2 + x_3 + x_5 \leq 1$$

cuts off the fractional point x', but is a valid inequality for $\mathcal{M}(G)$.

Answer to Exercise 1.3

a) We get $\dim(\mathcal{M}(G)) = m$ by the following argument: With $0 \in \mathcal{M}(G)$ and $u_e \in \mathcal{M}(G)$ for every $e \in E$ we have $m + 1$ affinely independent vectors contained in $\mathcal{M}(G)$, thus the matching polytope is fully-dimensional.

b) The face $F_e := \{x \in \mathcal{M}(G) : x_e = 0\}$ contains the vector 0 and also $u_{e'}$ for all $e' \in E \setminus \{e\}$, which are all affinely independent. This proves that F_e is a facet for every $e \in E$.

c) The node-edge incidence matrix S_G of the given graph is

$$S_G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

and $S_G \cdot x^* = (1,1,0,1)$, so the first, second and fourth inequality are active in x^*. Further, the inequalities $-x_2 \leq 0$ and $-x_3 \leq 0$ are also active. We thus obtain the cone of outer normals in x^* as

$$N_P(x^*) = \text{pos}\left\{ \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\0\\0\end{pmatrix} \right\}.$$

d) The point x' does obviously not satisfy the given inequality, so it is cut off by this inequality. To see why the inequality is valid for $\mathcal{M}(G)$, note that it corresponds to the odd cycle induced by the node set $\{2,3,4\}$. Summing up the inequalities of $S_G x \leq \mathbb{1}$ corresponding to these three nodes yields

$$x_1 + 2x_2 + 2x_3 + x_4 + 2x_5 \leq 3$$

and hence

$$2x_2 + 2x_3 + 2x_5 \leq 3,$$

as $x_e \geq 0$ for all edges $e \in E$. Dividing by 2 yields

$$x_2 + x_3 + x_5 \leq \frac{3}{2}.$$

To prove that an inequality is valid for the polytope $\mathcal{M}(G)$, it suffices to prove that is holds for all vertices of the polytope (all other points of the polytope then satisfy the inequality by a straightforward convex hull argument). So let $\bar{x}$ be some vertex of $\mathcal{M}(G)$, then in particular $\bar{x} \in \{0,1\}^5$ is integral. Of course, in that case the left hand side of the inequality is also integral, thus it holds for all vertices of $\mathcal{M}(G)$ if and only if the inequality

$$x_2 + x_3 + x_5 \leq \left\lfloor \frac{3}{2} \right\rfloor = 1$$

holds for all vertices of $\mathcal{M}(G)$. That completes the proof.