Technische Universität München, Zentrum Mathematik
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Discrete Optimization (MA 3502)

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Exercise Sheet 2

Exercise 2.1 (The Euclidean Algorithm) [8 credits]
Consider the following *Euclidean Algorithm*:

Input: $a, b \in \mathbb{N}$
Output: $\gcd(a, b)$
$k \leftarrow 0$;
$\sigma_0 \leftarrow \max\{a, b\}$;
$\sigma_1 \leftarrow \min\{a, b\}$;
repeat
  $k \leftarrow k + 1$;
  $\sigma_{k+1} \leftarrow \sigma_{k-1} - \left\lfloor \frac{\sigma_{k-1}}{\sigma_k} \right\rfloor \sigma_k$;
until $\sigma_{k+1} = 0$;
return σ_k

a) Let $a, b \in \mathbb{N}$ with $a \geq b \geq 1$. Show that $\left\lfloor \frac{a}{b} \right\rfloor \cdot b \geq \frac{a}{2}$.

b) Prove that the Euclidean Algorithm returns a correct solution in at most $\lfloor \log_2(a \cdot b) \rfloor + 1$ iterations.

c) Show that the Euclidean Algorithm can be modified to return $x_1, x_2 \in \mathbb{Z}$ with $a \cdot x_1 + b \cdot x_2 = \gcd(a, b)$.

   Hint: Construct two sequences α_k and β_k, $k \in \mathbb{N}$, with $a \cdot \alpha_k + b \cdot \beta_k = \sigma_k$.

d) Find two integers x_1, x_2 with $121x_1 + 19x_2 = 1$.

Exercise 2.2 [4 credits]
In Sudoku a 9×9 square is given, which is divided into nine 3×3 boxes. Some entries contain prescribed numbers from 1 to 9. The task is to fill the remaining entries such that within each row, column, and box every number from 1 to 9 appears exactly once.

Model the Sudoku task as an integer linear optimization problem.

Exercise 2.3 [6 credits]
Consider the Traveling Salesman Problem (TSP) on the complete directed graph $D_n = (V, A)$ on the vertex set $V = \{0, 1, \ldots, n-1\}$ with some distance function $d : A \to \mathbb{R}_{\geq 0}$. In the following, we will devise integer programming models for the TSP using variables $x_a \in \{0, 1\}$ for every arc $a \in A$ that indicate whether a is used in the optimal tour ($x_a = 1$) or not ($x_a = 0$).

Please turn over.

a) Consider the following integer linear program:

$$\min \sum_{(i,j)\in A} d_{ij}x_{ij}$$
$$\sum_{j\in V:(i,j)\in A} x_{ij} = 1 \quad \text{for all } i \in V$$
$$\sum_{i\in V:(i,j)\in A} x_{ij} = 1 \quad \text{for all } j \in V$$
$$x_{ij} \in \{0,1\} \quad \text{for all } (i,j) \in A$$

Show that this ILP is not sufficient to capture the TSP by providing a feasible solution to the ILP that does not correspond to a feasible TSP tour.

b) Can you add additional linear constraints to the above ILP such that the feasible set corresponds to the set of feasible TSP tours? How many variables and how many constraints do you need for your formulation?

c) Consider the following mixed integer linear program:

$$\min \sum_{(i,j)\in A} d_{ij}x_{ij}$$
$$\sum_{j\in V:(i,j)\in A} x_{ij} = 1 \quad \text{for all } i \in V$$
$$\sum_{i\in V:(i,j)\in A} x_{ij} = 1 \quad \text{for all } j \in V$$
$$u_i - u_j + (n-1)x_{ij} \leq n-2 \quad \text{for all } (i,j) \in A \text{ with } i,j \neq 0$$
$$x_{ij} \in \{0,1\} \quad \text{for all } (i,j) \in A$$
$$u_i \in \mathbb{R} \quad \text{for all } i \in V \setminus \{0\}$$

Show that the feasible set of this MILP corresponds to the set of feasible traveling salesman tours.

Exercise 2.4 [5 credits]

a) Let $a \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}$. Show that the linear diophantine equation

$$a^T x = \beta$$

has an integer solution $x \in \mathbb{Z}^n$ if and only if $\gcd(a_1, a_2, \ldots, a_n)$ divides β.

b) Show that the linear diophantine equation $121x_1 + 19x_2 = \beta$, has an integer solution $x \in \mathbb{Z}^2$ for every $\beta \in \mathbb{Z}$. Determine all solutions by the method discussed in the lectures, and interpret your result geometrically.