Discrete Optimization (MA 3502)

Prof. Dr. P. Gritzmann | Dipl.-Math. Viviana Ghiglione | Dr. M. Ritter

Exercise Sheet 3

Exercise 3.1 [6 credits]

Determine the complete set of integral solutions of the following diophantine equation system:

$$\begin{pmatrix} 6 & -6 & 9 \\ 3 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

Exercise 3.2 [5 credits]

Consider the lattice $L(A) := \{Ax : x \in \mathbb{Z}^3\}$ generated by the columns of

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 1 & 3 & 5 \end{pmatrix}.$$

a) Sketch (parts of) the lattice $L(A)$.

b) Prove or disprove the following statement: There are two linearly independent columns of A that already generate the lattice $L(A)$.

Exercise 3.3 [7 credits]

Let $A \in \mathbb{Z}^{m\times n}$ be a matrix of full row rank $b \in \mathbb{Z}^m$. Show that the system $Ax = b$ has an integral solution if and only if $y^T b \in \mathbb{Z}$ for each $y \in \mathbb{R}^m$ with $y^T A \in \mathbb{Z}^n$. (This is an integer version of Farkas' lemma.)

Exercise 3.4 [6 credits]

Let $A \in \mathbb{Z}^{m\times n}$ be a matrix of full row rank and $L(A) := \{Ax : x \in \mathbb{Z}^n\}$ the lattice generated by its columns. We define the *dual lattice* $L(A)^\perp$ by

$$L(A)^\perp := \left\{ y \in \mathbb{R}^m : y^T z \in \mathbb{Z} \text{ for all } z \in L(A) \right\}.$$

a) Show for a square, nonsingular matrix $A \in \mathbb{Z}^{n\times n}$ that $L(A)^\perp = L((A^{-1})^T)$. In other words, $L(A)^\perp$ is generated by the rows of A^{-1}.

b) Show for square, nonsingular matrix $A \in \mathbb{Z}^{n\times n}$ that $L(A)^{\perp\perp} = L(A)$.