**Discrete Optimization (MA 3502)**

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# Exercise Sheet 4

### Exercise 4.1

Consider the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 1 & 3 & 4 & 5 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Determine all vectors $b \in \mathbb{Z}^2$ for which there exists an integer solution $x \in \mathbb{Z}^4$ of this system, and determine for each such $b$ the set of solutions $L(b) := \{x \in \mathbb{Z}^4 : Ax = b\}$.

### Answer to Exercise 4.1

To determine the possible $b \in \mathbb{Z}^2$ for which the system is solvable, we first compute the Hermite normal form of $A$ and the associated unimodular transformation matrix $C$.

**Row 1** As operations we use (in this order) $S_1 = S_1 - 2S_2$, $S_3 = S_3 - S - 2$, $S_4 = S_4 - S_2$. Finally, we swap of columns 1 and 2.

**Row 2** As operations we use (in this order) $S_2 = S_2 + 5S_3$, $S_4 = S_4 - 2S_3$, $S_1 = S_1 - 3S_3$ and finally the swap of columns 2 and 3.

$$\left(\begin{array}{cccc} 4 & 2 & 2 & 2 \\ 1 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 3 & -5 & 1 & 2 \\ \hline 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 4 & -7 & -1 & 1 \\ -3 & 5 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 4 & -1 & -7 & 1 \\ -3 & 1 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Thus we obtain the Hermite normal form $H$ of $A$ and a corresponding unimodular transformation $C$ with $AC = H$:

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 4 & -1 & -7 & 1 \\ -3 & 1 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The system of equations $Ax = b$ has an integer solution, if and only if the system $A(Cy) = b$ has an integer solution, i. e. if and only if $Hy = b$ has an integer solution. Hence we need

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}b_1 \\ b_2 \end{pmatrix} \in \mathbb{Z}^2 \quad \text{and} \quad y_3, y_4 \in \mathbb{Z}.$$

which is the case if and only if $b_1$ is even and $b_2$ is an arbitrary integer.

For such a right hand side vector the set of solutions $L(b)$ can be determined by setting $y_1 := \frac{1}{2}b_1$, $y_2 := b_2$ and $y_3, y_4 \in \mathbb{Z}$ arbitrary:

$$L(b) = \left\{ C \begin{pmatrix} \frac{1}{2}b_1 \\ b_2 \\ y_3 \\ y_4 \end{pmatrix} : y_3, y_4 \in \mathbb{Z} \right\} = \begin{pmatrix} 0 \\ 2b_1 - b_2 \\ -\frac{3}{2}b_1 + b_2 \\ 0 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 1 \\ -7 \\ 5 \\ 0 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

**Exercise 4.2**

Let $P$ and $Q$ be polytopes in $\mathbb{R}^n$ where $n \geq 1$. Prove or disprove: $I(P \setminus Q) = I(P) \setminus I(Q)$.

**Answer to Exercise 4.2**

The statement is false. The integer hull of a polyhedron needs to be convex, while the difference of two convex sets does not need to be convex. In $\mathbb{R}^1$, take $P = \{x : -1 \leq x \leq 1\}$, $Q = \{0\}$. Then

$$I(P \setminus Q) = \operatorname{conv}\{-1, 1\} = [-1, 1] \neq I(P) \setminus I(Q) = \operatorname{conv}\{-1, 0, 1\} \setminus \{0\} = [-1, 1] \setminus \{0\}.$$

**Exercise 4.3**

Let $P \subset \mathbb{R}^2$ be defined through the following $\mathcal{H}$-presentation:

$$\begin{aligned} -\sqrt{2}x_1 + x_2 &\leq 0 \\ x_1 - \sqrt{2}x_2 &\leq 0 \end{aligned}$$

a) Sketch both $P$ and the integral points contained in $P$. Take a guess at the integer hull $I(P) := \operatorname{conv}(P \cap \mathbb{Z}^2)$ based on your sketch!

b) Show that $P = \operatorname{pos}\left\{(1, \sqrt{2})^T, (\sqrt{2}, 1)^T\right\}$ ($\mathcal{V}$-presentation).

c) Prove that $I(P) = \{0\} \cup \operatorname{int}(P)$. You may use the following result without proof: For every line in $\mathbb{R}^2$ with irrational slope there exist integer points arbitrarily close to (and on both sides of) the line.

d) Is $I(P)$ a polyhedron?

e) Let $Q := \left\{(-\frac{1}{2}, -\frac{1}{2})^T\right\} + P$. Show that the integer hull $I(Q)$ of $Q$ has an infinite number of vertices.

**Answer to Exercise 4.3**

a) The sketch in Figure 1 shows both $P$ and the integral points contained in $P$. For the integer hull, see the rest of this problem.

b) The two inequalities defining $P$ clearly intersect in 0, and as they do not define parallel lines, 0 is the only vertex of $P$. The outer normals of $P$ at 0 are $(-\sqrt{2}, 1)^T$ and $(1, -\sqrt{2})^T$, thus the extreme rays are perpendicular to these and pointing towards the positive orthand. This yields the desired $\mathcal{V}$-presentation

$$P = \operatorname{pos}\left\{ \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \right\}.$$

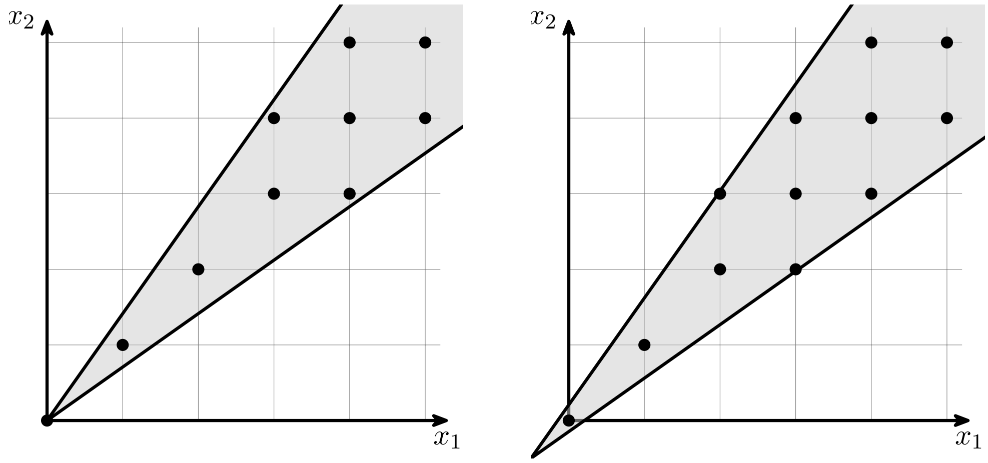


Figure 1: Illustration of $P$ and $Q$ and the integral points contained therein.

c) **"$\subset$":** We start by showing $I(P) \subset \{0\} \cup \operatorname{int}(P)$. Cleary, $I(P) \subset P$, thus it suffices to show that 0 is the only integer point lying on the boundary of $P$. Assume otherwise, then the extreme ray that contained such a point would have rational slope (since it also contains 0). This clearly contradicts the irrationality of $\sqrt{2}$, thus proving that 0 is in fact the only integral point on the boundary of $P$.

**"$\supset$":** We show the converse, i. e. $\{0\} \cup \operatorname{int}(P) \subset I(P)$, in two steps. Clearly, $0 \in I(P)$, hence we only need to consider some $x \in \operatorname{int}(P)$, $x \neq 0$.

i) If $\operatorname{pos}\{x\}$ contains a rational point, then it also contains some integer point $p$ (multiply with the denominator), and by multiplying with some suitably large $n \in \mathbb{N}$ we can assume that $x \in \operatorname{conv}\{0, np\}$ (meaning $x$ lies on the line segment connecting 0 and $np$). Since $\{0, np\} \subset P \cap \mathbb{Z}^2$ we have $\operatorname{conv}\{0, np\} \subset I(P)$ and thus $x \in I(P)$.

ii) If $\operatorname{pos}\{x\}$ contains no rational point, we denote the extreme rays of $P$ by $r_1 := \operatorname{pos}\left\{(1, \sqrt{2})^T\right\}$ and $r_2 := \left\{(\sqrt{2}, 1)^T\right\}$, respectively, and consider the rays $x + r_1$ and $x + r_2$. Note that $x + r_1 \subset \operatorname{int}(P)$, $x + r_2 \subset \operatorname{int}(P)$, and $\operatorname{pos}\{x\} \in \operatorname{pos}\{x + r_1, x + r_2\}$.

Using the statement in the hint, we know that there are integer points arbitrarily close to $x + r_1$ and $x + r_2$, respectively. Thus we can choose integer points $p_1, p_2$ that are close enough to $x + r_1$ and $x + r_2$ to still be contained in $\operatorname{int}(P)$ and such that they are on opposing sides of $\operatorname{pos}\{x\}$. Then, $\operatorname{pos}\{x\} \subset \operatorname{pos}\{p_1, p_2\} \subset I(P)$, showing $x \in I(P)$.

*Remark: Suppose all vertices of a polyhedron $P$ to be integer, and consider an LP over $P$ with a finite optimum. We know that the LP has a vertex solution, which means it has an integer solution, in this special case. Still, $P$ needs not to be equal to $I(P)$, as this exercise shows.*

d) The integer hull $I(P)$ is not a polyhedron. Assume otherwise, then $I(P)$ could be written as $I(P) = \operatorname{conv} V + \operatorname{pos} R$ with a finite set of vertices $V$ and of extreme rays $R$. However, this representation clearly is closed, in contrast to $I(P) = \{0\} \cup \operatorname{int}(P)$ as shown in the previous problem, a contradiction.

e) First note that $I(P) \neq I(Q)$, so the result of the preceding problem cannot simply be transferred. The reason for this is that the "surrounding cone" that defines $Q$ (and thus $I(Q)$) has been shifted such that its apex is not an integral point anymore.

It is easy to see that $p_0 := 0$ is a vertex of $I(Q)$, see Figure 1 for an illustration. Suppose that $I(Q)$ has only finitely many vertices $v_1, \dots, v_k$. Since $I(Q)$ is unbounded there must be an $n \in \mathbb{N}$ such that $v_n$ has only one neighboring vertex (the "last vertex at the verge of inifinity"). Let $r := \operatorname{pos}\left\{(1, \sqrt{2})^T\right\}$ and consider the rays $s_1 := -(\frac{1}{2}, \frac{1}{2}) + r$ and $s_2 := v_n + r$. Notice that $v_n$ is not lying on $s_1$, because otherwise that line would contain two rational points which would mean it had a rational slope. We again use the hint given for one of previous problems: There must be an integer point between $s_1$ and $s_2$, and if that was contained in $I(Q)$ then $v_n$ could not have been a vertex, a contradiction. Thus $I(Q)$ cannot be described by any finite number of vertices and extreme rays.