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Discrete Optimization (MA 3502)

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Exercise Sheet 5

Exercise 5.1

Which of the following statements are true and which are false? Give a proof or counterexample.

(a) If $A \in \mathbb{Z}^{n\times n}$ is totally unimodular, then all its eigenvalues are in $\{-1, 0, 1\}$

(b) If $A \in \mathbb{Z}^{d\times n}$ and $B \in \mathbb{Z}^{m\times n}$ are totally unimodular, then $\begin{pmatrix} A \\ B \end{pmatrix}$ is unimodular.

(c) If $A \in \mathbb{Z}^{n\times n}$ is totally unimodular and has rank n, then HNF$(A) = E_n$.

(d) If A and B are totally unimodular, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is also totally unimodular.

Answer to Exercise 5.1

(a) False. For instance, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is totally unimodular and 2 is an eigenvalue.

(b) False. For instance, $A = (1, 1)$ and $B = (1, -1)$ are totally unimodular, but $\begin{pmatrix} A \\ B \end{pmatrix}$ has determinant 2, and thus not totally unimodular.

(c) True. The determinant of HNF(A) is the same as that of A up to sign. Since A has rank n and is totally unimodular, it has determinant ± 1. Thus HNF(A) has determinant 1 (all entries on the diagonal must be positive, and the determinant is their product). Furthermore, since HNF(A) is integral, the values on the diagonal are all 1, and therefore the values elsewhere must be 0. Thus HNF$(A) = E_n$.
Alternatively: A is totally unimodular and full rank, hence it is unimodular. The matrix A^{-1} exists and it is unimodular as well. From the definition of HNF, we know that there exists an unimodular matrix $C \in \mathbb{Z}^n$ such that $AC =$HNF(A). Chosing $C = A^{-1}$, we get $AA^{-1} = E_n$ which is a matrix in HNF. Being the HNF unique, it must be HNF$(A) = E_n$.

(d) True. Any square submatrix C has the form $C = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$, where A' and B' are submatrices of A and B respectively. If these are not square, then C has determinant 0. If they are square, then $\det(C) = \det(A)\det(B) \in \{-1, 0, 1\}$.

Exercise 5.2

For the following matrices, determine whether they are unimodular or totally unimodular. If the matrices are not unimodular (respectively, totally unimodular), identify corresponding submatrices whose determinant is not in $\{-1, 0, +1\}$.

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Answer to Exercise 5.2

A_1 is unimodular (it has determinant 1), but not totally unimodular since

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

is a square submatrix with determinant 2 (take rows 2 and 3, and columns 1, and 3).

A_2 is not unimodular since

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

is a square submatrix of maximal size with determinant -2. For the same reason, A_2 is also not totally unimodular.

A_3 is unimodular and totally unimodular.

Exercise 5.3

Let $A \in \{-1, 0, +1\}^{m\times n}$. Show that if A is totally unimodular then the following holds:

a) Every regular submatrix of A has at least one row with an odd number of non-zero entries.

b) The sum over all entries in every square submatrix with even row and column sums is divisible by 4.

Answer to Exercise 5.3

Every $k \times k$-submatrix B of a totally unimodular matrix A is (by definition) again totally unimodular. Thus, by Theorem 3.3.2, there is for every $k \times k$-submatrix B of A a partition (I_1, I_2) of the columns b_i of B such that $\sum_{i\in I_1} b_i - \sum_{i\in I_2} b_i \in \{0, \pm 1\}^k$; in other words there exists an $x \in \{-1, +1\}^k$ such that $Bx = \{0, \pm 1\}^k$.

a) If every row of B contained an even number of non-zero entries, then Bx would contain only even entries. Since $Bx \in \{-1, 0, 1\}^k$ this would imply $Bx = 0$ and thus $x = 0$. However, this contradicts $x \in \{-1, +1\}^k$.

b) Consider a (square) submatrix B with row indices I and column indices J, and consider the (by Theorem 3.3.2) claimed partition (I_1, I_2) of I such that $\sum_{i\in I_1} b_i - \sum_{i\in I_2} b_i \in \{0, \pm 1\}^k$. In fact, the stronger property $\sum_{i\in I_1} b_i - \sum_{i\in I_2} b_i = 0$ holds, because the column sums of B yield even numbers. We therefore have

$$\sum_{i\in I_1} \sum_{j\in J} b_{ij} = \sum_{i\in I_2} \sum_{j\in J} b_{ij} =: 2s$$

$$\text{and thus} \quad \sum_{i\in I} \sum_{j\in J} b_{ij} = \sum_{i\in I_1} \sum_{j\in J} b_{ij} + \sum_{i\in I_2} \sum_{j\in J} b_{ij} = 2s + 2s = 4s,$$

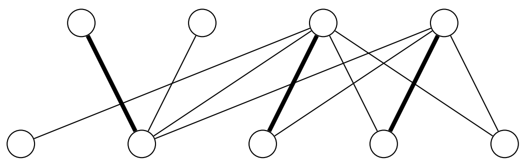
proving the claim.

Remark: That B should be a square submatrix is not really needed in (b), but it is stated since it can be computationally easier to check (than checking all submatrices).

Exercise 5.4 (The König-Egerváry-Theorem)

Let $G = (V, E)$ be a bipartite Graph. A node subset $U \subset V$ is called *node cover* of G if every edge in E is incident with a least one node in U. Prove that a node cover with a minimum number of nodes has the same cardinality as a matching with a maximum number of edges in G.

Use the theorem to prove that the following matching is maximum:



Answer to Exercise 5.4
Let $n := |V|$ and $m := |E|$ and let S_G be the node-edge incidence matrix of G. We consider the following ILP formulations for the minimum node cover problem and the maximum matching problem, respectively:

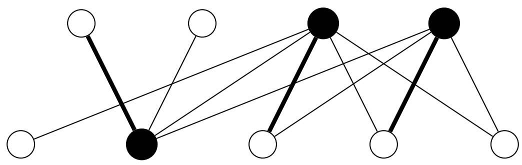
$$\begin{aligned} \min\ & \mathbb{1}_n^T y \\ & S_G^T y \geq \mathbb{1}_m \\ & y \geq 0 \\ & y \in \mathbb{Z}^n \end{aligned} \qquad\qquad \begin{aligned} \max\ & \mathbb{1}_m^T x \\ & S_G x \leq \mathbb{1}_n \\ & x \geq 0 \\ & x \in \mathbb{Z}^m \end{aligned}$$

As S_G is a totally unimodular matrix, the polyhedra defined by the above ILPs are already integer without explicitly requiring $x \in \mathbb{Z}^m$ or $y \in \mathbb{Z}^n$. Also, the respective LP relaxations are each other's duals. We may hence use the duality theorem to obtain equality of the objective values of both LPs. More precisely

$$\begin{aligned} & \min \left\{ \mathbb{1}_n^T y : S_G^T y \geq \mathbb{1}_n, y \in \mathbb{N}_0 \right\} \\ = & \min \left\{ \mathbb{1}_n^T y : S_G^T y \geq \mathbb{1}_n, y \geq 0 \right\} \\ = & \max \left\{ \mathbb{1}_m^T x : S_G x \leq \mathbb{1}_m, x \geq 0 \right\} \\ = & \max \left\{ \mathbb{1}_m^T x : S_G x \leq \mathbb{1}_m, x \in \mathbb{N}_0 \right\} \end{aligned}$$

That proves the König-Egerváry Theorem.

As an application, consider the matching given above and the following node cover:



Obviously, every edge is covered by one of the three marked nodes, so a minimum node cover has at most cardinality 3. As no maximum matching can be larger, the given matching of cardinality 3 is indeed a maximum matching (note this also proves that the node cover is minimum).

Exercise 5.5
Let $G = (V, E)$ be a graph and

$$P'_{\mathrm{M}}(G) := \left\{ x \in \mathbb{R}^{|E|} : x_e \geq 0 \text{ for all } e \in E, \text{ and } \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V \right\}$$

the fractional matching polytope over G. Show that $P_{\mathrm{M}}(G)$ is half-integral, i. e. the components of all extreme points are in $\left\{0, \frac{1}{2}, 1\right\}$.

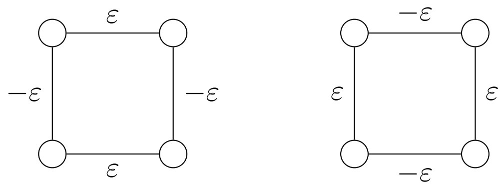


Figure 1

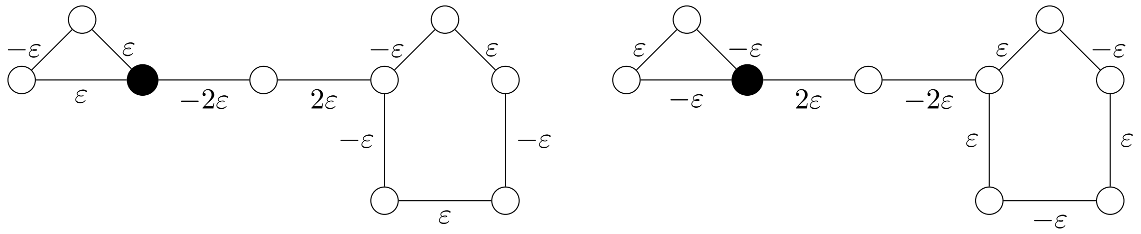


Figure 2

Hint: Show that every non half-integral extreme point $x \in P'_{\mathrm{M}}(G)$ can be expressed as convex combination of two points $y, z \in P'_{\mathrm{M}}(G)$.

Answer to Exercise 5.5

For the sake of contradiction, we assume that $x \in P'_{\mathrm{M}}(G)$ is an extreme point with at least one edge with $0 < x_e < 1$ and $x_e \neq 1/2$. This edge must have an adjacent edge that is also fractional; and as this argument can be iterated and the graph is finite, we obtain a cycle that consists of fractional edges.

We distinguish two cases:

Case 1: x contains a fractional cycle of even length On the cycle we alternatingly add and subtract a small $\varepsilon > 0$ from x and obtain y as shown on the left of Figure 1. We choose ε small enough such that $x_e - \varepsilon \geq 0$, i. e., such that y is feasible for P'_{M}. Analogously, we obtain z from x by subtracting and adding ε from x as shown on the right of Figure 1. We clearly have $x = (y+z)/2$, and thus x is a convex combination of two points $x \neq y$, hence x cannot be an extreme point.

Case 2: x contains no fractional cycle of even length but a fractional cycle of odd length As $x_e \neq 1/2$, the odd fractional cycle must contain a vertex adjacent to three fractional edges, otherwise we could not have $\sum_{e \in \delta(v)} x_e = 1$ for the vertex v closing the cycle. We follow the third edge (as seen on the left of Figure 2; see the bold vertex) and find a second fractional cycle (since G is finite). By our assumption, this must also be an odd cycle and it cannot share any edges with the first cycle (otherwise, the graph would also contain an even cycle obtained by simply ignoring the shared edges). If we now form y from x by adding/subtracting ε (and 2ε, respectively) as can be seen on the left of Figure 2 and if we form z analogously (see right of Figure 2), we obtain $y \neq z$ with $y, z \in P'_{\mathrm{M}}$ and $x = (y+z)/2$. An analogous construction works for an odd number of edges connecting the two cycles. Again, this contradicts that x is an extreme point of P'_{M}.