Technische Universität München, Zentrum Mathematik
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Discrete Optimization (MA 3502)

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Exercise Sheet 5

Exercise 5.1
Which of the following statements are true and which are false? Give a proof or counterexample.

(a) If $A \in \mathbb{Z}^{n\times n}$ is totally unimodular, then all its eigenvalues are in $\{-1, 0, 1\}$

(b) If $A \in \mathbb{Z}^{d\times n}$ and $B \in \mathbb{Z}^{m\times n}$ are totally unimodular, then $\begin{pmatrix} A \\ B \end{pmatrix}$ is unimodular.

(c) If $A \in \mathbb{Z}^{n\times n}$ is totally unimodular and has rank n, then $\mathrm{HNF}(A) = E_n$.

(d) If A and B are totally unimodular, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is also totally unimodular.

Exercise 5.2
For the following matrices, determine whether they are unimodular or totally unimodular. If the matrices are not unimodular (respectively, totally unimodular), identify corresponding submatrices whose determinant is not in $\{-1, 0, +1\}$.

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

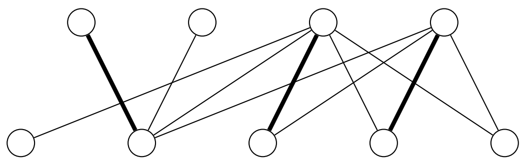
Exercise 5.3
Let $A \in \{-1, 0, +1\}^{m\times n}$. Show that if A is totally unimodular then the following holds:

a) Every regular submatrix of A has at least one row with an odd number of non-zero entries.

b) The sum over all entries in every square submatrix with even row and column sums is divisible by 4.

Exercise 5.4 (The König-Egerváry-Theorem)
Let $G = (V, E)$ be a bipartite Graph. A node subset $U \subset V$ is called *node cover* of G if every edge in E is incident with a least one node in U. Prove that a node cover with a minimum number of nodes has the same cardinality as a matching with a maximum number of edges in G.

Use the theorem to prove that the following matching is maximum:



Please turn over.

Exercise 5.5

Let $G = (V, E)$ be a graph and

$$P'_{\mathrm{M}}(G) := \left\{ x \in \mathbb{R}^{|E|} : x_e \geq 0 \text{ for all } e \in E, \text{ and } \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V \right\}$$

the fractional matching polytope over G. Show that $P_{\mathrm{M}}(G)$ is half-integral, i. e. the components of all extreme points are in $\left\{0, \frac{1}{2}, 1\right\}$.

Hint: Show that every non half-integral extreme point $x \in P'_{\mathrm{M}}(G)$ can be expressed as convex combination of two points $y, z \in P'_{\mathrm{M}}(G)$.