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# Discrete Optimization (MA 3502)

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## Exercise Sheet 6

*All problems marked with a star will be discussed in the tutorials (but you should nevertheless try those on your own). All other problems should preferably be prepared at home so that you can ask questions in the tutorials if you encounter any difficulties with these problems.*

### Exercise 6.1

Recall Corollary 3.3.3 from the lecture:

Let $A = (a_{ij}) \in \{-1, 0, +1\}^{m\times n}$ be such that each column of $A$ contains at most two nonzero entries. Then $A$ is totally unimodular if and only inf there exists a partition $(I_1, I_2)$ of the row indices $[m]$ with the property

$$a_{i_1,j} \cdot a_{i_2,j} < 0 \quad \Leftrightarrow \quad (\{i_1, i_2\} \subset I_1 \text{ or } \{i_1, i_2 \subset I_2\})$$

for all $i_1, i_2 \in [m]$ with $i_1 \neq i_2$ and $a_{i_1,j}, a_{i_2,j} \neq 0$.

Describe of a polynomial time algorithm which tests whether a given matrix $A$ fulfills the conditions of Corollary 3.3.3.

*Hint:* Try to construct a graph that is bipartite if and only if the conditions of the corollary are met.

### Answer to Exercise 6.1

Clearly, it can checked in polynomial time whether the columns of $A$ have at most two non-zero entries. Thus assuming this to be true, we construct a graph $G$ which is is bipartite if and only if the partition as described in Corollary 3.3.3 is possible. The idea is to construct a so-called *conflict graph*.

We assign one vertex to each row of $A$. If a column of $A$ contains two 1 entries or two $-1$ entries we connect the vertices of the corresponding rows with an edge, thus forcing those row nodes to be in different parts of the partition if the graph is bipartite. If a column contains one 1 and one $-1$ entry we add an additional vertex to the graph and connect this new vertex with the vertices corresponding to the $-1$ and 1 entries, thus forcing the row nodes to be in the same part of the partition for a bipartite graph. (Alternatively, the two vertices could be identified as being one vertex.)

Obviously, every partition of the rows of $A$ as in Corollary 3.3.3 leads to a partition of the vertices in the constructed graph which shows that it is bipartite and vice versa. Bipartiteness of a graph can be checked in polynomial time: For instance, apply breadth-first-search and color the layers alternatingly; the graph is not bipartite if and only if a conflict arises.

### Exercise 6.2

Show that $A \in \{-1, 0, 1\}^{m\times n}$ is totally unimodular if and only if the polytope

$$P := \{x \in \mathbb{R}^n : a \leq Ax \leq b, c \leq x \leq d\}$$

is integral for all $a, b \in \mathbb{Z}^m$ and $c, d \in \mathbb{Z}^n$.

### Answer to Exercise 6.2

The system $a \leq Ax \leq b$, $c \leq x \leq d$ can be expressed as

$$\left\{ x \in \mathbb{R}^n : \begin{pmatrix} A \\ -A \\ -E_n \end{pmatrix} \leq \begin{pmatrix} b \\ -a \\ -c \end{pmatrix}, x \leq d \right\}.$$

The claim now follows using Remark 3.3.1: $A$ is totally unimodular if and only if

$$\begin{pmatrix} A \\ -A \\ -E_n \end{pmatrix}$$

is totally unimodular (this is by Remark 3.3.1); this is (by Remark 3.2.12) equivalent to

$$\begin{pmatrix} A \\ -A \\ -E_n \\ E_n \end{pmatrix}$$

being unimodular; and this is equivalent (by Proposition 3.2.5) to the assertion that all vertices of

$$\left\{ x \in \mathbb{R}^n : \begin{pmatrix} A \\ -A \\ -E_n \end{pmatrix} \leq \begin{pmatrix} b \\ -a \\ -c \end{pmatrix}, x \leq d \right\}$$

are integer for every integer vector $(a, -b, -c)^T$ and $d$.

**Exercise 6.3** (Consecutive Ones Matrices)
A matrix $A \in \{0,1\}^{m \times n}$ has the *consecutive ones property* (along columns), if after a possible reordering of the rows of $A$ the 1-entries appear consecutively in each column. Prove that any matrix with the consecutive ones property is totally unimodular.

**Answer to Exercise 6.3**
Let $A$ be a consecutive ones matrix. Let us first consider the case that no reordering of the rows is necessary to obtain consecutive ones in each column. We will use 3.3.2 to prove the claim: Let $I \subset [m]$, $I \neq \emptyset$ and assume $I = \{i_1, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Set $I_1 := \{i_1, i_3, i_5 \ldots\}$ and $I_2 := \{i_2, i_4, i_6, \ldots\}$, then

$$\sum_{i \in I_1} a_i^T - \sum_{i \in I_2} a_i^T \in \{-1, 0, 1\}^T :$$

- Columns with an even number of ones in the rows in $I$ yield a value of 0, as $I_1, I_2$ contain an equal number of ones from that column due to the consecutive ones property.
- Columns with an odd number of ones in the rows in $I$ yield a value of $+1$ or $-1$, depending on whether there is one more 1-entry in $I_1$ or in $I_2$.

Hence, by 3.3.2, the matrix is totally unimodular.

A reordering of the rows of $A$ corresponds to a reordering of the rows of the submatrices of $A$. As reordering rows only changes the sign, but not the absolute value of the determinant of a matrix, total unimodularity is not affected by changing the order of the rows of $A$, thus the claim also holds for general consecutive ones matrices.

$^\star$**Exercise 6.4** (The Chvátal-Gomory closure)

Let

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & 0 \\ 1 & 2 \\ 1 & -2 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}. \right\}$$

The (first) *Chvátal-Gomory closure* of $P$, denoted by $P'$ or $P^{(1)}$, is obtained by adding all possible round-off cuts to the polyhedron $P$. Repeating that process yields the second Chvátal-Gomory closure $P^{(2)} := (P^{(1)})'$, and so on. The *Chvátal-Gomory rank* of $P$ is the smallest integer $k$ such that $P^{(k)} = I(P)$.

a) Show that the Chvátal-Gomory closure of $P$ is given by

$$P^{(1)} = \operatorname{conv} \left\{ (0,0)^T, (0,1)^T, (1/2, 1/2)^T \right\}.$$

b) Show that the Chvátal-Gomory closure of $P^{(1)}$ is $P^{(2)} = I(P) = \operatorname{conv} \left\{ (0,0)^T, (0,1)^T \right\}$.

c) Let $k \in \mathbb{N}$ and

$$Q = \operatorname{conv} \left\{ (0,0)^T, (0,1)^T, (k, 1/2)^T \right\}.$$

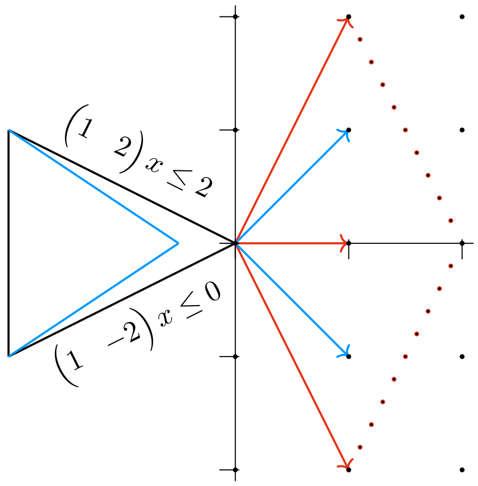
Show: $Q^{(2k-1)} \neq I(Q)$ and $Q^{(2k)} = I(Q) = I(P)$ by proving $Q^{(i)} = \operatorname{conv} \left\{ (0,0)^T, (0,1)^T, (k - i/2, 1/2) \right\}$.,

**Answer to Exercise 6.4**

a) It suffices to construct the Hilbert basis of the cone of the normal vectors of the unique fractional vertex and to add the corresponding (irredundant) cuts. The Hilbert basis is

$$\left\{ (1,2)^T, (1,1)^T, (1,0)^T, (1,-1)^T, (1,-2)^T \right\}.$$

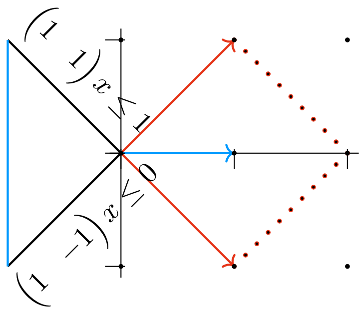
Thus the vectors $(1,1)^T$ and $(1,-1)^T$ yield irredundant cuts. The claim follows.




b) Again it suffices to construct the Hilbert basis of the cone of the normal vectors of the unique fractional vertex and to add the corresponding (irredundant) cuts. The Hilbert basis is

$$\left\{ (1,1)^T, (1,0)^T, (1,-1)^T \right\}$$

The vector $(1,0)^T$ yields the desired cut to obtain $P_2 = P_I$.




c) To simplify the notation a little, we define $R^{(i)} := Q^{(2k-i)}$. We show by induction over $i$ that

$$R^{(i)} = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\frac{i}{2}\\ 1/2\end{pmatrix}\right\},$$

which yields the desired result.

We already proved that $R^{(0)} = (R^{(1)})'$, thus it remains to show that $R^{(i-1)} = (R^{(i)})'$. Obviously, $R^{(i)}$ can be described by the following $\mathcal{H}$-presentation:

$$\begin{aligned} x_1 + ix_2 &\le i \\ x_1 - ix_2 &\le 0 \\ -x_1 &\le 0 \end{aligned}$$

Only the first two inequalities are active in the unique fractional vertex $x^{(i)} = (i/2, 1/2)^T$, thus we consider the Hilbert basis spanned by their respective outer normals. This Hilbert basis is

$$H^{(i)} = \left\{\begin{pmatrix}1\\j\end{pmatrix} : -i \le j \le i\right\},$$

producing the round-off cuts

$$x_1 + jx_2 \le \left\lfloor \frac{i}{2} + \frac{j}{2} \right\rfloor.$$

This only yields a proper cut for odd values of $i + j$, more precisely cuts of the form $x_1 + jx_2 \le \frac{i+j-1}{2}$.

Inserting $x^{(i-1)} = (i-1/2, 1/2)$ into these inequalities yields

$$\frac{i-1}{2} + \frac{j}{2} = \frac{i+j-1}{2},$$

thus $x^{(i-1)}$ fulfuills all these conditions and therefore $R(i-1) \subset (R^{(i)})'$. For $j \in \{i-1, -i+1\}$ we obtain the constraints

$$\begin{aligned} x_1 + (i-1)x_2 &\le i - 1 \\ x_1 - (i-1)x_2 &\le 0, \end{aligned}$$

which, together with $-x_1 \le 0$, give the $\mathcal{H}$-presentation of $R^{(i-1)}$, thus $(R^{(i)})' \subset R^{(i-1)}$. Together, this shows the desired equality.