Technische Universität München, Zentrum Mathematik
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Discrete Optimization (MA 3502)

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Exercise Sheet 6

All problems marked with a star will be discussed in the tutorials (but you should nevertheless try those on your own). All other problems should preferably be prepared at home so that you can ask questions in the tutorials if you encounter any difficulties with these problems.

Exercise 6.1
Recall Corollary 3.3.3 from the lecture:

Let $A = (a_{ij}) \in \{-1, 0, +1\}^{m\times n}$ be such that each column of A contains at most two nonzero entries. Then A is totally unimodular if and only inf there exists a partition (I_1, I_2) of the row indices $[m]$ with the property

$$a_{i_1,j} \cdot a_{i_2,j} < 0 \quad \Leftrightarrow \quad (\{i_1, i_2\} \subset I_1 \text{ or } \{i_1, i_2 \subset I_2\})$$

for all $i_1, i_2 \in [m]$ with $i_1 \neq i_2$ and $a_{i_1,j}, a_{i_2,j} \neq 0$.

Describe of a polynomial time algorithm which tests whether a given matrix A fulfills the conditions of Corollary 3.3.3.

Hint: Try to construct a graph that is bipartite if and only if the conditions of the corollary are met.

Exercise 6.2
Show that $A \in \{-1, 0, 1\}^{m\times n}$ is totally unimodular if and only if the polytope

$$P := \{x \in \mathbb{R}^n : a \leq Ax \leq b, c \leq x \leq d\}$$

is integral for all $a, b \in \mathbb{Z}^m$ and $c, d \in \mathbb{Z}^n$.

Exercise 6.3 (Consecutive Ones Matrices)
A matrix $A \in \{0, 1\}^{m\times n}$ has the *consecutive ones property* (along columns), if after a possible reordering of the rows of A the 1-entries appear consecutively in each column.Prove that any matrix with the consecutive ones property is totally unimodular.

Please turn over.

***Exercise 6.4** (The Chvátal-Gomory closure)
Let

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & 0 \\ 1 & 2 \\ 1 & -2 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} = \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}. \right\}$$

The (first) *Chvátal-Gomory closure* of P, denoted by P' or $P^{(1)}$, is obtained by adding all possible round-off cuts to the polyhedron P. Repeating that process yields the second Chvátal-Gomory closure $P^{(2)} := (P^{(1)})'$, and so on. The *Chvátal-Gomory rank* of P is the smallest integer k such that $P^{(k)} = I(P)$.

a) Show that the Chvátal-Gomory closure of P is given by

$$P^{(1)} = \operatorname{conv}\left\{ (0,0)^T, (0,1)^T, (1/2, 1/2)^T \right\}.$$

b) Show that the Chvátal-Gomory closure of $P^{(1)}$ is $P^{(2)} = I(P) = \operatorname{conv}\left\{ (0,0)^T, (0,1)^T \right\}$.

c) Let $k \in \mathbb{N}$ and

$$Q = \operatorname{conv}\left\{ (0,0)^T, (0,1)^T, (k, 1/2)^T \right\}.$$

Show: $Q^{(2k-1)} \neq I(Q)$ and $Q^{(2k)} = I(Q) = I(P)$ by proving $Q^{(i)} = \operatorname{conv}\left\{ (0,0)^T, (0,1)^T, (k - i/2, 1/2) \right\}.$,