Technische Universität München, Zentrum Mathematik
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Discrete Optimization (MA 3502)

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Exercise Sheet 7

All problems marked with a star will be discussed in the tutorials (but you should nevertheless try those on your own). All other problems should preferably be prepared at home so that you can ask questions in the tutorials if you encounter any difficulties with these problems.

*Exercise 7.1

Consider the polyhedron $P = \{x \in \mathbb{R}^2 : Ax \leq b\}$ given by the following $\mathcal{H}$- and $\mathcal{V}$-presentation, respectively:

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -3 & -4 \\ -1 & 1 \\ 4 & 6 \\ 4 & -10 \end{pmatrix} x \leq \begin{pmatrix} -8 \\ 2 \\ 27 \\ 3 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}, \begin{pmatrix} \frac{9}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \right\}$$

a) Draw a sketch of both P and its integer hull $I(P)$.

b) Can you determine some proper cut for $I(P)$ just by "staring" at the inequalities?

c) Consider the vertex $x^* = (3/2, 7/2)^T$ of P. Compute a Gomory cut at x^* with respect to the first component.

d) Using $B = \{2, 3\}$ as a starting basis corresponding to x^*, use the dual simplex algorithm and the Gomory cutting plane algorithm to compute the optimal solution to the following ILP:

$$\begin{aligned} \max\ & x_2 \\ & Ax \leq b \\ & x \in \mathbb{Z}^2 \end{aligned}$$

Answer to Exercise 7.1

a) Figure 1 shows a sketch of both P and its integer hull.

b) The last two inequalities have only even coefficients on the left hand side, but an odd right hand side. Thus, for each integral vector x, the left hand side of both inequalities will be even, and hence the right hand side may be rounded down to the nearest even integer. Formally, we can divide both inequalities by 2 and then apply rounding. This procedure yields the cuts

$$\begin{aligned} 2x_1 + 3x_2 &\leq \left\lfloor \frac{27}{2} \right\rfloor = 13 \\ 2x_1 - 5x_2 &\leq \left\lfloor \frac{3}{2} \right\rfloor = 1 \end{aligned}$$

Both cuts are illustrated in red in Figure 1.

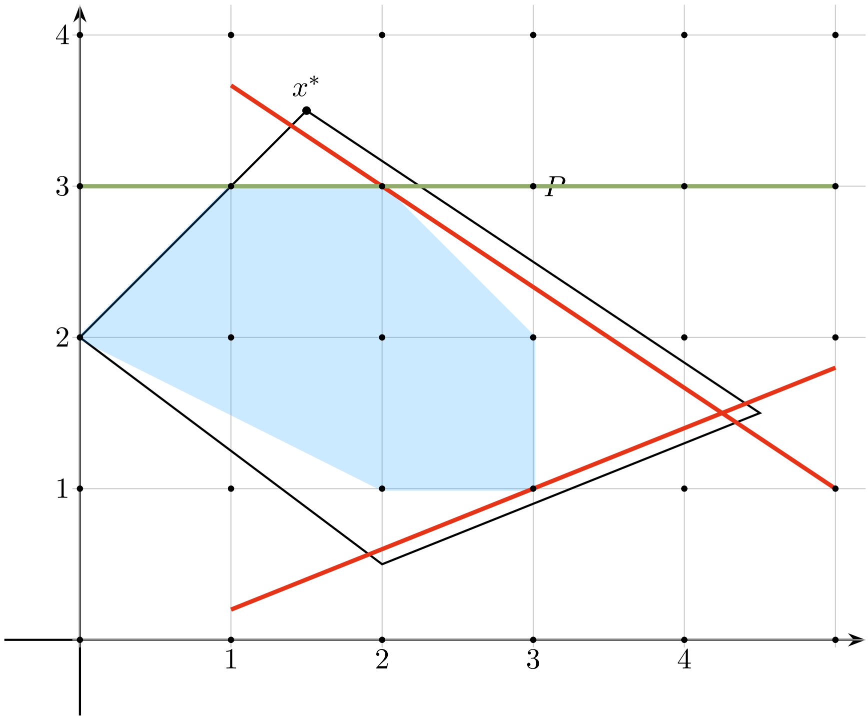

Figure 1: The polytope P and its integer hull $I(P)$.

c) Substituting x^* into the inequalities yields $Ax^* = (-37/2, 2, 27, -29)^T$, thus the second and third inequality are active in x^*. The unique basis corresponding to x^* is therefore $B = \{2, 3\}$ which yields

$$A_B = \begin{pmatrix} -1 & 1 \\ 4 & 6 \end{pmatrix} \quad \text{and} \quad A_B^{-1} = \frac{1}{10} \begin{pmatrix} -6 & 1 \\ 4 & 1 \end{pmatrix}.$$

As we need to cut with respect to the first component, our integer Gomory vector is $v = (1, 0)^T$ and hence

$$y_B^* = \left\langle (A_B^{-1})^T v \right\rangle = \left\langle \begin{pmatrix} -6/10 \\ 1/10 \end{pmatrix} \right\rangle = \begin{pmatrix} 4/10 \\ 1/10 \end{pmatrix}$$

$$y_N^* = 0$$

$$q^* = A^T y = \begin{pmatrix} -3 & -1 & 4 & 4 \\ -4 & 1 & 6 & -10 \end{pmatrix} \begin{pmatrix} 0 \\ 4/10 \\ 1/10 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left\lfloor (q^*)^T x^* \right\rfloor = \left\lfloor \frac{7}{2} \right\rfloor = 3$$

This results in the Gomory cut $x_2 \leq 3$ illustrated in green in Figure 1.

To illustrate the idea behind Gomory cuts, consider the Hilbert zonotope of P at the point x^* as depicted in Figure 2 and imagine a partition of the plane into translated copies of the Hilbert zonotope. The vector v is drawn in red, it points at one of the integer points in a translated copy of the Hilbert zonotope. This uniquely identifies one integer point in the Hilbert zonotope, and the vector pointing at this point in the original Hilbert zonotope is the cut vector q^* for the Gomory cut.

d) For the dual simplex algorithm, we need the problem to be of the form min $c^T x$, $Ax = b$, $x \geq 0$. In our case, there are two possible ways to achieve this: We could either transform directly, introducing slack variables and using variable splitting to get sign-restricted variables. Or we could just use the dual of our LP which already has the correct form min $b^T z$, $A^T z = c$, $z \geq 0$ (we will use z as the dual variable to avoid confusion with the Gomory cut computation later on):

$$\begin{aligned} \min\ & -8z_1 + 2z_2 + 27z_3 + 3z_4 \\ & \begin{pmatrix} -3 & -1 & 4 & 4 \\ -4 & 1 & 6 & -10 \end{pmatrix} z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & z \geq 0 \end{aligned}$$

(Note that we are dualizing the LP, not the ILP – duality theory does not work for ILPs in general, so we cannot just solve the integer dual directly!)

The primal basis $B = \{2, 3\}$ translates into the dual basic variables $\{z_2, z_3\}$. Recall the dual simple txableau:

$$\begin{array}{c|c} -b^T z & b - b_B^T (A_B^T)^{-1} A \\ \hline (A_B^T)^{-1} c & (A_B^T)^{-1} A^T \end{array}$$

We compute A_B^T and its inverse first and get

$$A_B^T = \begin{pmatrix} -1 & 4 \\ 1 & 6 \end{pmatrix}, \qquad (A_B^T)^{-1} = \frac{1}{10} \begin{pmatrix} -6 & 4 \\ 1 & 1 \end{pmatrix}.$$

This yields the following dual simplex tableau:

$$\begin{array}{c|cccc} -\frac{35}{10} & \frac{21}{2} & 0 & 0 & 32 \\ \hline \frac{4}{10} & \frac{2}{10} & 1 & 0 & -\frac{64}{10} \\ \frac{1}{10} & -\frac{7}{10} & 0 & 1 & -\frac{6}{10} \end{array}$$

As all reduced costs are nonnegative, the solution is already optimal. The corresponding primal solution is $x^* = (3/2, 7/2)^T$, so we do not have an integral solution yet. A Gomory cut could now be applied with respect to the objective function or any component, as all three values are fractional. Let us start with an objective cut:

$$\begin{aligned} y_B^* &= \left\langle (A_B^T)^{-1} c \right\rangle = \left\langle \frac{1}{10} \begin{pmatrix} -6 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \begin{pmatrix} \frac{4}{10} \\ \frac{1}{10} \end{pmatrix} \\ y_N^* &= 0 \\ q^* = A^T y^* &= \begin{pmatrix} -3 & -1 & 4 & 4 \\ -4 & 1 & 6 & -10 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{4}{10} \\ \frac{1}{10} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

This yields the Gomory cut $x_2 \leq \left\lfloor \frac{7}{2} \right\rfloor = 3$.

Adding this cut to the primal problem unfortunately renders our primal basis infeasible, so we would have to start over the simplex algorithm from phase I. However, the dual basis $\{z_2, z_3\}$ will stay feasible, the cut corresponds to an additional variable (and thus column) in the dual problem. Our new dual then looks like this:

$$\begin{aligned} \min\ & -8z_1 + 2z_2 + 27z_3 + 3z_4 + 3z_5 \\ & \begin{pmatrix} -3 & -1 & 4 & 4 & 0 \\ -4 & 1 & 6 & -10 & 1 \end{pmatrix} z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & z \geq 0 \end{aligned}$$

Adding the new column to the dual simplex tableau can be done easily by just computing the product of $(A_B^T)^{-1}$ and the new column and changing the reduced cost line accordingly. This yields the new tableau

$$\begin{array}{c|ccccc} -\frac{35}{10} & \frac{21}{2} & 0 & 0 & 32 & \overbrace{-\frac{1}{2}}^{3-\frac{2\cdot 4+27\cdot 1}{10}} \\ \hline \frac{4}{10} & \frac{2}{10} & 1 & 0 & -\frac{64}{10} & \frac{4}{10} \\ \frac{1}{10} & -\frac{7}{10} & 0 & 1 & -\frac{6}{10} & \frac{1}{10} \end{array}$$

The newly added fifth column is now our pivot column and we have to compute the minimum of the quotient of the left hand side and the tableau matrix entry in that column for each row where the matrix entry is positive to find the pivot element. Here, we get

$$\frac{4/10}{4/10} = 1$$
$$\text{and} \quad \frac{1/10}{1/10} = 1,$$

thus we may choose either entry as pivot element. Let us use the first entry (i. e., z_5 enters the basis and z_2 leaves it) and compute the new simplex tableau

$$\begin{array}{c|ccccc} -3 & \frac{43}{4} & \frac{5}{4} & 0 & 24 & 0 \\ \hline 1 & \frac{1}{2} & \frac{5}{2} & 0 & -16 & 1 \\ 0 & -\frac{15}{20} & -\frac{5}{20} & 1 & 1 & 0 \end{array}$$

The solution is optimal now, with a dual solution of $z = (0, 0, 1, 0, 0)^T$ and dual basis $\{z_3, z_5\}$. However, as we care about the primal solution, we may not be done yet (remember that a dual integral solution does not guarantee primal integrality), we still have to compute the primal solution. With $B = \{3, 5\}$ as the current basis we get

$$A_B = \begin{pmatrix} 4 & 6 \\ 0 & 1 \end{pmatrix}, \quad A_B^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -6 \\ 0 & 4 \end{pmatrix}, \quad b_B = \begin{pmatrix} 27 \\ 3 \end{pmatrix}$$

and hence $x^* = (\frac{9}{4}, 3)^T$. The cut and the new primal optimal point can be seen in Figure 3.

That means we are not yet done, as x^* still has non-integral components. Let's do another Gomory cut, this time with respect to the first component (we do not have another choice this time):

$$y_B^* = \left\langle (A_B^T)^{-1} c \right\rangle = \left\langle \frac{1}{4} \begin{pmatrix} 1 & 0 \\ -6 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix}$$

$$y_N^* = 0$$

$$q^* = A^T y^* = \begin{pmatrix} -3 & -1 & 4 & 4 & 0 \\ -4 & 1 & 6 & -10 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1/4 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This yields the Gomory cut $x_1 + 2x_2 \leq \left\lfloor \frac{33}{4} \right\rfloor = 8$, it is illustrated in Figure 4. The cut changes the dual problem and thus the simplex tableau as follows:

$$\begin{aligned} \min \; & -8z_1 + 2z_2 + 27z_3 + 3z_4 + 3z_5 + 8z_6 \\ & \begin{pmatrix} -3 & -1 & 4 & 4 & 0 & 1 \\ -4 & 1 & 6 & -10 & 1 & 2 \end{pmatrix} z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & z \geq 0 \end{aligned}$$

Adding the new column to the dual simplex tableau can be done as before by computing the product of $(A_B^T)^{-1}$ and the new column and changing the reduced cost line accordingly:

$$\begin{array}{c|cccccc} -3 & \frac{43}{4} & \frac{5}{4} & 0 & 24 & 0 & \overbrace{-\frac{1}{4}}^{=8-\frac{27\cdot 1+3\cdot 2}{4}} \\ \hline 1 & \frac{1}{2} & \frac{5}{2} & 0 & -16 & 1 & \frac{1}{4} \\ 0 & -\frac{15}{20} & -\frac{5}{20} & 1 & 1 & 0 & \frac{1}{2} \end{array}$$

The newly added sixth column is the only possible pivot column and we have to compute the minimum of the quotient of the left hand side and the tableau matrix entry in that column for each row where the matrix entry is positive to find the pivot element. We get

$$\begin{aligned} \frac{1}{1/4} &= 4 \\ \text{and} \quad \frac{0}{1/2} &= 0, \end{aligned}$$

thus the second row contains our pivot element. That means when z_6 enters the basis, the variable z_3 will at the same time leave it. Let us compute the new simplex tableau using Gaussian elimination:

$$\begin{array}{c|cccccc} -3 & \frac{83}{8} & \frac{9}{8} & \frac{1}{2} & \frac{49}{2} & 0 & 0 \\ \hline 1 & \frac{7}{8} & \frac{11}{8} & -\frac{1}{2} & -\frac{33}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & 2 & 2 & 0 & 1 \end{array}$$

Note that we are doing a degenerate simplex step here: We are changing the basis, but stay in fact at the same dual solution. Our new dual optimal basis is $\{z_5, z_6\}$, and for the corresponding primal basis the matrix A_B is

$$A_B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad b_B = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

and hence $x^* = (2,3)^T$. This is an integer solution, so the procedure terminates with $x^* = (2,3)^T$ as the optimal integral solution to our original ILP.

Exercise 7.2

With the Gomory cutting plane algorithm we compute solutions of an ILP as follows:

Let P_0 the feasible set of an LP-relaxation of the ILP. Iterate over i: Compute the optimal solution x^i over P_i and check if it is integral. If yes, we are done. Otherwise add a Gomory cut to P_i with v choosen as

a) the objective c, if $c^T x^i$ is fractional,

b) a unit vector u^k, $k \in [n]$, if $c^T x^i$ and x^i_j, $j \leq k-1$ are integral, but x^i_k is fractional.

The new polyhedron is denoted by P_{i+1}.

Use the Gomory cutting plane algorithm to compute an optimal solution of the follwing ILP:

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 10, \\ & x_1 - 2x_2 \leq 0 \end{aligned}$$

Draw a sketch illustrating the situation in each step, compute the optimal solution via the dual simplex method, and determine the basis of the current optimal primal solutions by considering the respective active constraints.

Answer to Exercise 7.2

a) The optimal solution of the LP in the first iteration is $x^1 = (20/7, 10/7)^T$ (see also Exercise 2.1). The corresponding (primal) basis is $B_1 = \{1,2\}$. The objective function value $c^T x^1 = 130/7$ is fractional. We therefore add a Gomory cut with respect to c:

$$\left(c^T - \left\lfloor c^T A_{B_1}^{-1} \right\rfloor A_{B_1}\right) x \leq \left\lfloor c^T x^1 \right\rfloor - \left\lfloor c^T A_{B_1}^{-1} \right\rfloor A_{B_1} x^1,$$

i.e.,

$$2\xi_1 + 2\xi_2 \leq 8.$$

Note that $A_{B_1}^{-1} = ((M^{B_1})^{-1})^T$ with $((M^{B_1})^{-1})^T$ from Exercise 2.1.

b) The optimal solution of the LP in the second iteration is $x^2 = (8/3, 4/3)^T$ (see also Exercise 2.1). The corresponding (primal) basis is $B_2 = \{2,3\}$. The objective function value $c^T x^2 = 52/3$ is fractional. We therefore add a Gomory cut with respect to c:

$$\left(c^T - \left\lfloor c^T A_{B_2}^{-1} \right\rfloor A_{B_2}\right) x \leq \left\lfloor c^T x^2 \right\rfloor - \left\lfloor c^T A_{B_2}^{-1} \right\rfloor A_{B_2} x^2,$$

i.e.,

$$\xi_1 - \xi_2 \leq 1.$$

Note that $A_{B_2}^{-1} = ((M^{B_2})^{-1})^T$ with $((M^{B_2})^{-1})^T$ from Exercise 2.1.

c) The optimal solution of the LP in the third iteration is $x^3 = (5/2, 3/2)^T$ (see also Exercise 2.1). The corresponding (primal) basis is $B_3 = \{3, 4\}$. The objective function value $c^T x^3 = 17$ is integral. As u_1 is fractional, we add a Gomory cut with respect to u_1 (by the way, here the same cut would be generated if we selected the cut with respect to u_2):

$$\left(u_1^T - \left\lfloor u_1^T A_{B_3}^{-1} \right\rfloor A_{B_3}\right) x \leq \left\lfloor u_1^T x^3 \right\rfloor - \left\lfloor u_1^T A_{B_3}^{-1} \right\rfloor A_{B_3} x^3$$

i.e.,

$$\xi_1 \leq 2.$$

Note that $A_{B_3}^{-1} = ((M^{B_3})^{-1})^T$ with $((M^{B_3})^{-1})^T$ from Exercise 2.1.

d) The optimal solution of the LP in the fourth iteration is $x^4 = (2, 2)^T$ (see also Exercise 2.1). We thus have arrived at the optimal integer solution; the algorithm terminates. The optimal objective function value is 16.

Remark: It is worth noting that three constraints are active in x^4, namely constraint 1, 3, and 5. The dual basis for the corresponding dual optimal solution $y^4 = (1, 0, 0, 0, 3)$ is, however, $\{1, 5\}$. While $\{3, 5\}$ is also an optimal primal basis (yielding the optimal x^4), it is not an optimal dual basis. That we could reuse the inverse matrices computed in the dual simplex method for the A_B^{-1} computations is thus only coincidental.

For graphical sketches see the solutions to Exercise 2.1(a).

Exercise 7.3

Let $C := \text{pos}\{s_1, s_2\}$ with $s_1 := (1, 2)^T$ and $s_2 := (2, 1)^T$.

a) Show that

$$C \cap \mathbb{Z}^2 = \left\{ \begin{pmatrix} i \\ i \end{pmatrix} + j \begin{pmatrix} 1 \\ 0 \end{pmatrix} : i \in \mathbb{N}_0,\ j = 0, \ldots, i \right\} \cup \left\{ \begin{pmatrix} i \\ i \end{pmatrix} + j \begin{pmatrix} 0 \\ 1 \end{pmatrix} : i \in \mathbb{N}_0\ j = 0, \ldots, i \right\}.$$

b) Show by induction over i that $\{(1, 2)^T, (2, 1)^T, (1, 1)^T\}$ form a Hilbert basis of C.

c) Conclude that $\{(1, 2)^T, (2, 1)^T, (1, 1)^T\}$ form a minimal Hilbert basis of C.

Answer to Exercise 7.3

(a) For $i \in \mathbb{N}_0$ and $j = 0, \ldots, i$ we have

$$\begin{pmatrix} i+j \\ i \end{pmatrix} = \tfrac{1}{3}(i-j)s_1 + \tfrac{1}{3}(i+2j)s_2, \qquad (1)$$

$$\begin{pmatrix} i \\ i+j \end{pmatrix} = \tfrac{1}{3}(i+2j)s_1 + \tfrac{1}{3}(i-j)s_2, \qquad (2)$$

and thus since $i - j \geq 0$ and $i + 2j$ we conclude that these vectors are in $C \cap \mathbb{Z}^2$. For the converse, observe first that for $(k, l)^T \in \mathbb{C} \cap \mathbb{Z}^2$ we must have $k, l \in \mathbb{N}_0$. For $k \geq l$ we can write $(k, l)^T = (i + j, j)^T$ with $i, j \in \mathbb{N}_0$, and from Eq. (1) we have seen that we must have $j \leq i$ (note that the coefficients of s_1 and s_2 are unique since $\{s_1, s_2\}$ form a basis of $\mathbb{R}^2$). The remaining case $k < l$ follows from this by noting the symmetry of C along $(1,1)^T$. Alternatively, one can write

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} i' - j' \\ i' - j' + j' \end{pmatrix} = \frac{1}{3}(i' - j' + 2j')s_1 + \frac{1}{3}(i' - 2j')s_2,$$

with $i', j' \in \mathbb{N}_0$ and $i' - j' \geq 0$, which for $(i'-j', i'-j'+j')^T \in C$ implies $i' - 2j' \geq 0 \Leftrightarrow i' - j' \geq j'$; and this (by setting $i := i' - j'$, $j := j'$) is a vector as in Eq. (2).

(b) For $i \in \mathbb{N}_0$, let $C(i) := \{(i+j, i)^T : j = 0, \ldots, i\}$ and $C'(i) := \{(i, i+j)^T : j = 0, \ldots, i\}$.

For $i = 0$, we clearly have $(0,0)^T = 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot \mathbb{1}$, and thus $v \in C(0) \cup C'(0)$ implies that v can be expressed as non-negative integer linear combination of elements in $\{s_1, s_2, \mathbb{1}\}$.

Now, consider $v \in C(i+1)$. Since $C(i+1) = C(i) \cup \{(i+1+j, i+1)^T : j = 0, \ldots, i+1\}$ we can assume by induction hypothesis that $v \in \{(i+1+j, i+1)^T : j = 0, \ldots, i+1\}$. However, for $0 \leq j \leq i$ we have

$$\begin{pmatrix} i+1+j \\ i+1 \end{pmatrix} = \begin{pmatrix} i \\ i \end{pmatrix} + j \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and for $j = i+1$ we have

$$\begin{pmatrix} 2i+2 \\ i+1 \end{pmatrix} = (i+1) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus v can be expressed as non-negative integer linear combination of elements in $\{s_1, s_2, \mathbb{1}\}$.

We proceed similarly for $v \in C'(i+1)$ (or consider the symmetry of C along $(1,1)^T$). Since $C'(i+1) = C'(i) \cup \{(i+1, i+1+j)^T : j = 0, \ldots, i+1\}$ we can assume by induction hypothesis that $v \in \{(i+1, i+1+j)^T : j = 0, \ldots, i+1\}$. However, for $0 \leq j \leq i$ we have

$$\begin{pmatrix} i+1 \\ i+1+j \end{pmatrix} = \begin{pmatrix} i \\ i \end{pmatrix} + j \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and for $j = i+1$ we have

$$\begin{pmatrix} i+1 \\ 2i+2 \end{pmatrix} = (i+1) \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus v can be expressed as non-negative integer linear combination of elements in $\{s_1, s_2, \mathbb{1}\}$.

In summary, $\{s_1, s_2, \mathbb{1}\}$ is a Hilbert basis of C.

(c) We show that none of the vectors s_1, s_2, and $\mathbb{1}$ can be expressed as a positive integer linear combination of other points in $C \cap \mathbb{Z}^2$ since this implies that every Hilbert basis of C needs to contain s_1, s_2, and $\mathbb{1}$ (and this shows with (b) that $\{s_1, s_2, \mathbb{1}\}$ is a minimal Hilbert basis of C).

The only candidiates for expressing s_1 as positive integer linear combination of other points in $C \cap \mathbb{Z}^2$ are $(0,0)^T$ and $(1,1)^T$ (the other points have a larger first component); and clearly there are no $\lambda_1, \lambda_2 \in \mathbb{N}_0$ with $s_1 = \lambda_1(0,0)^T + \lambda_2(1,1)^T$. In the same way we see that we cannot express s_2 as a positive integer linear combination of $(0,0)^T$ and $(1,1)^T$; and, of course, $(1,1)^T$ cannot be expressed as a positive integer linear combination of $(0,0)^T$.

Exercise 7.4

Let $C := \mathbb{R}$ be the cone of real numbers. Show that for each $k \in \mathbb{N}, k \geq 2$ there is a minimal Hilbert basis of C that has cardinality $k+1$.

Hint: Consider k different prime numbers and define $k+1$ distinct products from these to get a suitable Hilbert basis.

Answer to Exercise 7.4

Let $\pi_1, \dots, \pi_k$ be different prime numbers, and define

$$q_0 := -\prod_{j=1}^{k} \pi_j \qquad \wedge \qquad q_i := \prod_{\substack{j=1 \\ j \neq i}}^{k} \pi_j \qquad (i = 1, \dots, k).$$

Then it holds $\gcd(q_1, \dots, q_k) = 1$, and from Lemma 2.2.2 it follows

$$\sum_{i=1}^{k} \mathbb{Z} q_i = \mathbb{Z}.$$

Let $l \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k \in \mathbb{Z}$ with $\lambda_1, \dots, \lambda_l < 0$, $\lambda_{l+1}, \dots, \lambda_k > 0$ and $\lambda \in \mathbb{N}$, so that it holds:

$$\sum_{i=1}^{k} \lambda_i q_i = 1 \quad \wedge \quad \lambda \pi_1 \geq |\lambda_1| \wedge \ldots \wedge \lambda \pi_l \geq |\lambda_l|$$

We get

$$\sum_{i=1}^{l} (\lambda_i + \lambda \pi_i) q_i + \sum_{i=l+1}^{k} \lambda_i q_i = 1 - \lambda l q_0$$

Hence

$$1 = \lambda l q_0 + \sum_{i=1}^{l} (\lambda_i + \lambda \pi_i) q_i + \sum_{i=l+1}^{k} \lambda_i q_i,$$

which means that 1 can be generated as a positive integer combination of $q_0, q_1, \dots, q_k$.
As $-1 = q_0 + (-q_0 - 1)$, and $-q_0 - 1 > 0$, it follows

$$-1 = q_0 + (-q_0 - 1) \cdot 1 = q_0 + (-q_0 - 1)\Big(\lambda l q_0 + \sum_{i=1}^{l} (\lambda_i + \lambda \pi_i) q_i + \sum_{i=l+1}^{k} \lambda_i q_i\Big)$$

Thus

$$\sum_{i=0}^{k} \mathbb{N}_0 q_i = \mathbb{Z}.$$

We have proved that $\{q_0, q_1, \dots, q_k\}$ is a Hilbert basis of C. In order to show that it is minimal, it is sufficient to notice that the elements of every subset of $\{q_0, q_1, \dots, q_k\}$ have a non trivial common divisor. Then, again from Lemma 2.2.2, we conclude that $\{q_0, q_1, \dots, q_k\}$ is a minimal Hilbert Basis of C.

Example: $k = 2$, define $\pi_1 := 2$ and $\pi_2 := 3$ and

$$q_0 := -6 \qquad q_1 := 2 \qquad q_2 := 3,$$

and it holds with $\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda = 1$ (respectively $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda = 1$)

$$1 = q_0 + 2q_1 + q_2 \quad \wedge \quad -1 = q_0 + q_1 + q_2.$$

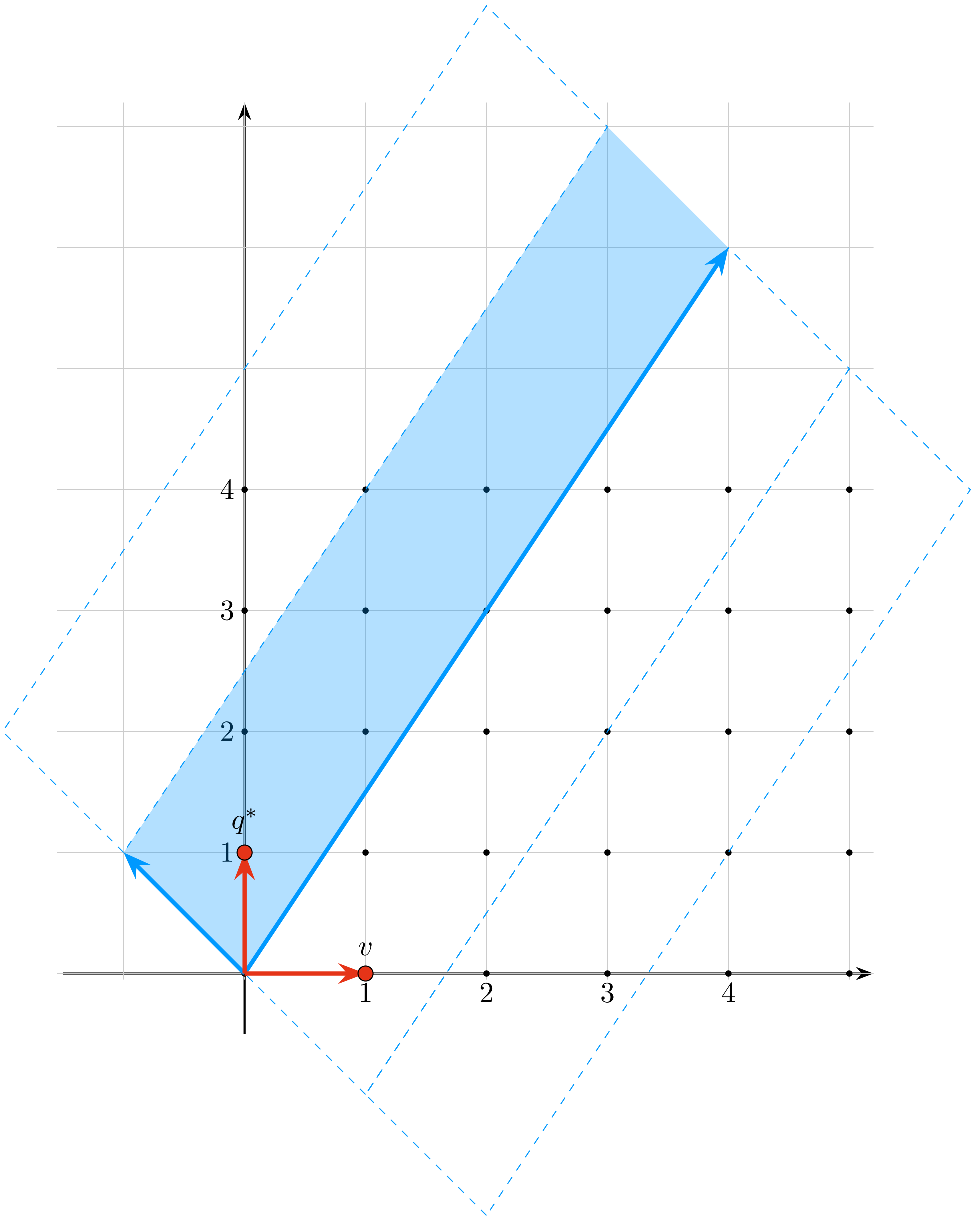

Figure 2: The Hilbert zonotope corresponding to x^*.

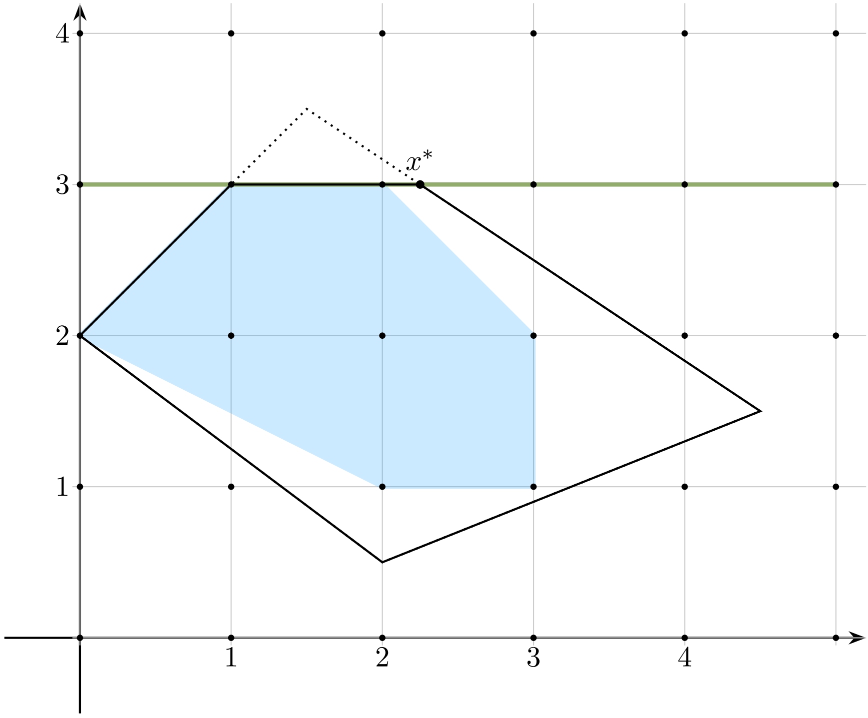

Figure 3: The first Gomory cut.

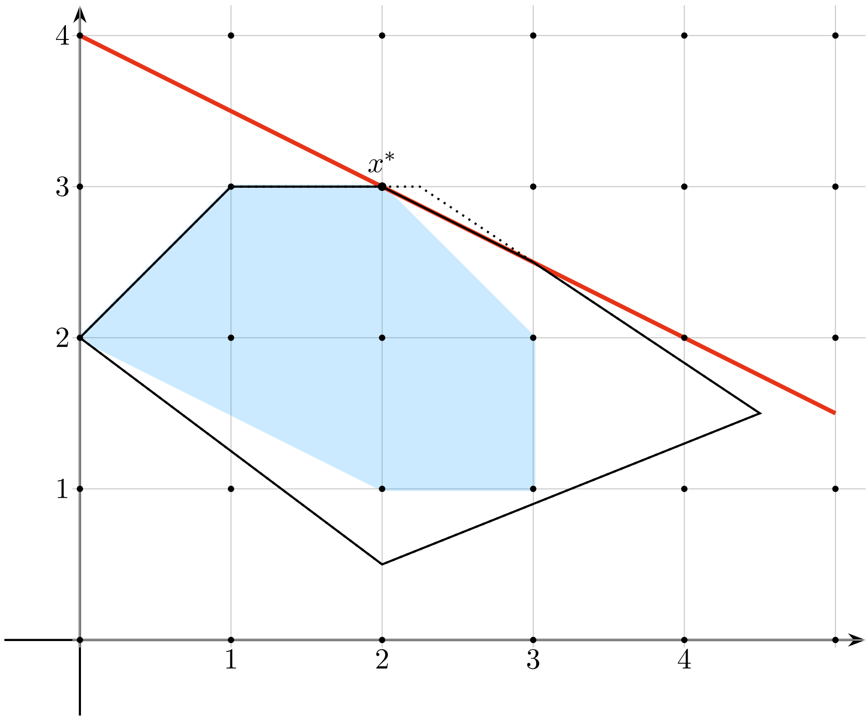

Figure 4: The second Gomory cut.