Technische Universität München, Zentrum Mathematik
Lehrstuhl für Angewandte Geometrie und Diskrete Mathematik

Discrete Optimization (MA 3502)

Prof. Dr. P. Gritzmann | Dipl.-Math. Viviana Ghiglione | Dr. M. Ritter

Exercise Sheet 7

All problems marked with a star will be discussed in the tutorials (but you should nevertheless try those on your own). All other problems should preferably be prepared at home so that you can ask questions in the tutorials if you encounter any difficulties with these problems.

*Exercise 7.1

Consider the polyhedron $P = \{x \in \mathbb{R}^2 : Ax \leq b\}$ given by the following $\mathcal{H}$- and $\mathcal{V}$-presentation, respectively:

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -3 & -4 \\ -1 & 1 \\ 4 & 6 \\ 4 & -10 \end{pmatrix} x \leq \begin{pmatrix} -8 \\ 2 \\ 27 \\ 3 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}, \begin{pmatrix} \frac{9}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \right\}$$

a) Draw a sketch of both P and its integer hull $I(P)$.

b) Can you determine some proper cut for $I(P)$ just by "staring" at the inequalities?

c) Consider the vertex $x^* = (3/2, 7/2)^T$ of P. Compute a Gomory cut at x^* with respect to the first component.

d) Using $B = \{2, 3\}$ as a starting basis corresponding to x^*, use the dual simplex algorithm and the Gomory cutting plane algorithm to compute the optimal solution to the following ILP:

$$\begin{aligned} \max\ & x_2 \\ & Ax \leq b \\ & x \in \mathbb{Z}^2 \end{aligned}$$

Exercise 7.2

With the Gomory cutting plane algorithm we compute solutions of an ILP as follows:

Let P_0 the feasible set of an LP-relaxation of the ILP. Iterate over i: Compute the optimal solution x^i over P_i and check if it is integral. If yes, we are done. Otherwise add a Gomory cut to P_i with v choosen as

a) the objective c, if $c^T x^i$ is fractional,

b) a unit vector u^k, $k \in [n]$, if $c^T x^i$ and x^i_j, $j \leq k-1$ are integral, but x^i_k is fractional.

The new polyhedron is denoted by P_{i+1}.

Use the Gomory cutting plane algorithm to compute an optimal solution of the follwing ILP:

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 10, \\ & x_1 - 2x_2 \leq 0 \end{aligned}$$

Draw a sketch illustrating the situation in each step, compute the optimal solution via the dual simplex method, and determine the basis of the current optimal primal solutions by considering the respective active constraints.

Exercise 7.3

Let $C := \text{pos}\{s_1, s_2\}$ with $s_1 := (1,2)^T$ and $s_2 := (2,1)^T$.

a) Show that

$$C \cap \mathbb{Z}^2 = \left\{ \begin{pmatrix} i \\ i \end{pmatrix} + j \begin{pmatrix} 1 \\ 0 \end{pmatrix} : i \in \mathbb{N}_0,\ j = 0, \dots, i \right\}$$
$$\cup \left\{ \begin{pmatrix} i \\ i \end{pmatrix} + j \begin{pmatrix} 0 \\ 1 \end{pmatrix} : i \in \mathbb{N}_0\ j = 0, \dots, i \right\}.$$

b) Show by induction over i that $\{(1,2)^T, (2,1)^T, (1,1)^T\}$ form a Hilbert basis of C.

c) Conclude that $\{(1,2)^T, (2,1)^T, (1,1)^T\}$ form a minimal Hilbert basis of C.

Exercise 7.4

Let $C := \mathbb{R}$ be the cone of real numbers. Show that for each $k \in \mathbb{N}, k \geq 2$ there is a minimal Hilbert basis of C that has cardinality $k+1$.

Hint: Consider k different prime numbers and define $k+1$ distinct products from these to get a suitable Hilbert basis.