An Extension of the Blow-up Lemma to arrangeable graphs *

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The Blow-up Lemma established by Komlós, Sárközy, and Szemerédi in 1997 is an important tool for the embedding of spanning subgraphs of bounded maximum degree. Here we prove several generalisations of this result concerning the embedding of a-arrangeable graphs, where a graph is called a-arrangeable if its vertices can be ordered in such a way that the neighbours to the right of any vertex v have at most a neighbours to the left of v in total. Examples of arrangeable graphs include planar graphs and, more generally, graphs without a K_s-subdivision for constant s. Our main result shows that a-arrangeable graphs with maximum degree at most $\sqrt{n}/\log n$ can be embedded into corresponding systems of super-regular pairs. This is optimal up to the logarithmic factor.

We also present two applications. We prove that any large enough graph G with minimum degree at least $\left(\frac{r-1}{r}+\gamma\right)n$ contains an F-factor of every a-arrangeable r-chromatic graph F with at most ξn vertices and maximum degree at most $\sqrt{n}/\log n$, as long as ξ is sufficiently small compared to $\gamma/(ar)$. This extends a result of Alon and Yuster [J. Combin. Theory Ser. B 66(2),

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269–282, 1996]. Moreover, we show that for constant p the random graph $\mathcal{G}(n,p)$ is universal for the class of a-arrangeable n-vertex graphs H of maximum degree at most $\xi n/\log n$, as long as ξ is sufficiently small compared to p/a.

1 Introduction

The last 15 years have witnessed an impressive series of results guaranteeing the presence of spanning subgraphs in dense graphs. In this area, the so-called *Blow-up Lemma* has become one of the key instruments. It emerged out of a series of papers by Komlós, Sárközy, and Szemerédi (see e.g. [15, 16, 17, 18, 19, 20, 21]) and asserts, roughly spoken, that we can find bounded degree spanning subgraphs in ε-regular pairs. It was used for determining, among others, sufficient degree conditions for the existence of F-factors, Hamilton paths and cycles and their powers, spanning trees and triangulations, and graphs of sublinear bandwidth in graphs, digraphs and hypergraphs (see the survey [24] for an excellent overview of these and related achievements). In this way, the Blow-up Lemma has reshaped extremal graph theory.

However, with very few exceptions, the embedded spanning subgraphs H considered so far came from classes of graphs with constant maximum degree, because the Blow-up Lemma requires the subgraph it embeds to have constant maximum degree. In fact, the Blow-up Lemma is usually the only reason why the proofs of the above mentioned results only work for such subgraphs.

The central purpose of this paper is to overcome this obstacle. We shall provide extensions of the Blow-up Lemma that can embed graphs whose degrees are allowed to grow with the number of vertices. These versions require that the subgraphs we embed are arrangeable.[1] We will formulate them in the following and subsequently present some applications.

Blow-up Lemmas. We first introduce some notation. Let G, H and R be graphs with vertex sets $V(G)$, $V(H)$, and $V(R) = \{1,\dots,r\} =: [r]$. For $v \in V(G)$ and $S, U \subseteq V(G)$ we define $N(v,S) := N(v) \cap S$ and $N(U,S) = \bigcup_{v\in U} N(v,S)$. Let $A, B \subset V(G)$ be non-empty and disjoint, and let $\varepsilon, \delta \in [0,1]$. The *density* of the pair (A,B) is defined to be $d(A,B) := e(A,B)/(|A||B|)$. The pair (A,B) is *ε-regular*, if $|d(A,B) - d(A',B')| \le \varepsilon$ for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge \varepsilon|A|$ and $|B'| \ge \varepsilon|B|$. An ε-regular pair (A,B) is called *(ε,δ)-regular*, if $d(A,B) \ge \delta$ and *(ε,δ)-super-regular*, if $|N(v,B)| \ge \delta|B|$ for all $v \in A$ and $|N(v,A)| \ge \delta|A|$ for all $v \in B$. We say that H has an *R-partition* $V(H) = X_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} X_r$, if for every edge $xy \in E(H)$ there are distinct $i, j \in [r]$ with $x \in X_i$, $y \in X_j$ and $ij \in E(R)$. G has a *corresponding (ε,δ)-super-regular R-partition* $V(G) = V_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_r$, if $|V_i| = |X_i| =: n_i$ for all $i \in [r]$ and every pair (V_i, V_j) with $ij \in E(R)$ is (ε,δ)-super-regular. In this case R is also called the *reduced graph* of the super-regular partition. Moreover, these partitions are *balanced* if $n_1 \le n_2 \le \dots \le n_r \le$

[1] We remark that it was already suggested in [13] to relax the maximum degree constraint to arrangeability.

$n_1 + 1$. They are *κ-balanced* if $n_j \le \kappa n_i$ for all $i, j \in [r]$. The partition classes V_i are also called *clusters*.

With this notation, a simple version of the Blow-up Lemma of Komlós, Sárközy, and Szemerédi [17] can now be formulated as follows.

Theorem 1 (Blow-up Lemma [17])
Given a graph R of order r and positive parameters δ, Δ, there exists a positive $\varepsilon = \varepsilon(r, \delta, \Delta)$ such that the following holds. Suppose that H and G are two graphs with the same number of vertices, where $\Delta(H) \le \Delta$ and H has a balanced R-partition, and G has a corresponding (ε, δ)-super-regular R-partition. Then there exists an embedding of H into G.

We remark that Rödl and Ruciński [28] gave a different proof for this result. In addition, Komlós, Sárközy, and Szemerédi [18] gave an algorithmic proof.

Our first result replaces the restriction on the maximum degree of H in Theorem 1 by a restriction on its arrangeability. This concept was first introduced by Chen and Schelp in [6].

Definition 2 (a-arrangeable)
*Let a be an integer. A graph is called a-*arrangeable *if its vertices can be ordered as $(x_1, \dots, x_n)$ in such a way that $\big|N\big(N(x_i, \mathrm{Right}_i), \mathrm{Left}_i\big)\big| \le a$ for each $1 \le i \le n$, where $\mathrm{Left}_i = \{x_1, x_2, \dots, x_i\}$ and $\mathrm{Right}_i = \{x_{i+1}, x_{i+2}, \dots, x_n\}$.*

Obviously, every graph H with $\Delta(H) \le a$ is $(a^2 - a + 1)$-arrangeable. Other examples for arrangeable graphs are planar graphs: Chen and Schelp showed that planar graphs are 761-arrangeable [6]; Kierstead and Trotter [12] improved this to 10-arrangeable. In addition, Rödl and Thomas [29] showed that graphs without K_s-subdivision are s^8-arrangeable. On the other hand, even 1-arrangeable graphs can have unbounded degree (e.g. stars).

Theorem 3 (Arrangeable Blow-up Lemma)
Given a graph R of order r, a positive real δ and a natural number a, there exists a positive real $\varepsilon = \varepsilon(r, \delta, a)$ such that the following holds. Suppose that H and G are two graphs with the same number of vertices, where H is a-arrangeable, $\Delta(H) \le \sqrt{n}/\log n$ and H has a balanced R-partition, and G has a corresponding (ε, δ)-super-regular R-partition. Then there exists an embedding of H into G.

Komlós, Sárközy, and Szemerédi proved that the Blow-up Lemma allows for the following strengthenings that are useful in applications. We allow the clusters to differ in size by a constant factor and we allow certain vertices of H to restrict their image in G to be taken from an a priori specified set of linear size. However, in contrast to the original Blow-up Lemma, we need to be somewhat more restrictive about the image restrictions: We still allow linearly many vertices in each cluster to have image restrictions, but now only a constant number of different image restrictions is permissible in each cluster (we shall show in Section 5 that this is best possible). In the following, we state an extended version of the Blow-up Lemma that makes this precise.

Theorem 4 (Arrangeable Blow-up Lemma, full version)
For all $C, a, \Delta_R, \kappa \in \mathbb{N}$ and for all $\delta, c > 0$ there exist $\varepsilon, \alpha > 0$ such that for every integer r there is n_0 such that the following is true for every $n \geq n_0$. Assume that we are given

(a) *a graph R of order r with $\Delta(R) < \Delta_R$,*
(b) *an a-arrangeable n-vertex graph H with maximum degree $\Delta(H) \leq \sqrt{n}/\log n$, together with a κ-balanced R-partition $V(H) = X_1 \dot\cup \ldots \dot\cup X_r$,*
(c) *a graph G with a corresponding (ε, δ)-super-regular R-partition $V(G) = V_1 \dot\cup \ldots \dot\cup V_r$ with $|V_i| = |X_i| =: n_i$ for every $i \in [r]$,*
(d) *for every $i \in [r]$ a set $S_i \subseteq X_i$ of at most $|S_i| \leq \alpha n_i$* image restricted vertices*, such that $|N_H(S_i) \cap X_j| \leq \alpha n_j$ for all $ij \in E(R)$,*
(e) *and for every $i \in [r]$ a family $\mathcal{I}_i = \{I_{i,1}, \ldots, I_{i,C}\} \subseteq 2^{V_i}$ of permissible* image restrictions*, of size at least $|I_{i,j}| \geq c n_i$ each, together with a mapping $I\colon S_i \to \mathcal{I}_i$, which assigns a permissible image restriction to each image restricted vertex.*

Then there exists an embedding $\varphi\colon V(H) \to V(G)$ such that $\varphi(X_i) = V_i$ and $\varphi(x) \in I(x)$ for every $i \in [r]$ and every $x \in S_i$.

As we shall show, the upper bound on the maximum degree of H in Theorem 4 is optimal up to the log-factor (see Section 5). However, if we require additionally that every $(a+1)$-tuple of G has a big common neighbourhood then this degree bound can be relaxed to $o(n/\log n)$.

Theorem 5 (Arrangeable Blow-up Lemma, extended version)
Let $a, \Delta_R, \kappa \in \mathbb{N}$ and $\iota, \delta > 0$ be given. Then there exist $\varepsilon, \xi > 0$ such that for every r there is $n_0 \in \mathbb{N}$ such that the following holds for every $n \geq n_0$.

Assume that we are given a graph R of order r with $\Delta(R) < \Delta_R$, an a-arrangeable n-vertex graph with $\Delta(H) \leq \xi n/\log n$, together with a κ-balanced R-partition, and a graph G with a corresponding (ε, δ)-super-regular R-partition $V = V_1 \dot\cup \ldots \dot\cup V_r$. Assume that in addition for every $i \in [r]$ every tuple $(u_1, \ldots, u_{a+1}) \subseteq V \setminus V_i$ of vertices satisfies $|\bigcap_{j \in [a+1]} N_G(u_j) \cap V_i| \geq \iota |V_i|$. Then there exists an embedding of H into G.

Again, the degree bound of $\xi n/\log n$ for H in Theorem 5 is optimal up to the constant factor. The same degree bound can be obtained if we do require H only to be an almost spanning subgraph, even if the additional condition on $(a+1)$-tuples from Theorem 5 is dropped again.

Theorem 6 (Arrangeable Blow-up Lemma, almost spanning version)
Let $\mu > 0$ and assume that we have exactly the same setup as in Theorem 4, but with $\Delta(H) \leq \xi n/\log n$ instead of the maximum degree bound given in (b), where ξ is sufficiently small compared to all other constants. Fix an a-arrangeable ordering of H, let X'_i be the first $(1-\mu)n_i$ vertices of X_i in this ordering, and set $H' := H[X'_1 \cup \cdots \cup X'_r]$.

Then there exists an embedding $\varphi\colon V(H') \to V(G)$ such that $\varphi(X'_i) \subseteq V_i$ and $\varphi(x) \in I(x)$ for every $i \in [r]$ and every $x \in S_i \cap X'_i$.

Let us point out that one additional essential difference between these three versions of the Blow-up Lemma and Theorem 1 concerns the order of the quantifiers: the regularity ε

that we require only depends on the maximum degree Δ_R of the reduced graph R, but *not* on the number of the vertices in R. Sometimes this is useful in applications. Clearly, we can reformulate our theorems to match the original order of quantifiers of Theorem 1; the lower bound on n_0 can be omitted in this case.

Applications. To demonstrate the usefulness of these extensions of the Blow-up Lemma, we consider two example applications that can now be derived in a relatively straightforward manner. At the end of this section we are going to mention a few further applications that are more difficult and will be proven in separate papers.

Our first application concerns F-factors in graphs of high minimum degree. This is a topic which is well investigated for graphs F of *constant* size. For a graph F on f vertices, an *F-factor* in a graph G is a collection of vertex disjoint copies of F in G such that all but at most $f-1$ vertices of G are covered by these copies of F.

A classical theorem by Hajnal and Szemerédi [9] states that each n-vertex graph G with minimum degree $\delta(G) \geq \frac{r-1}{r}n$ has a K_r-factor. Alon and Yuster [3] considered arbitrary graphs F and showed that, if r denotes the chromatic number of F, every sufficiently large graph G with minimum degree $\delta(G) \geq (\frac{r-1}{r}+\gamma)n$ contains an F-factor. This was improved upon by Komlós, Sárközy, and Szemerédi [21], who replaced the linear term γn in the degree bound by a constant $C = C(F)$; and by Kühn and Osthus [25], who, inspired by a result of Komlós [14], determined the precise minimum degree threshold for every constant size F up to a constant.

In contrast to the previous results we consider graphs F whose size may grow with the number of vertices n of the host graph G. More precisely, we allow graphs F of size linear in n. To prove this result, we use Theorem 4 (see Section 6) and hence we require that F is a-arrangeable and has maximum degree at most $\sqrt{n}/\log n$.

Theorem 7
For every a, r and $\gamma > 0$ there exist n_0 and $\xi > 0$ such that the following is true. Let G be any graph on $n \geq n_0$ vertices with $\delta(G) \geq (\frac{r-1}{r}+\gamma)n$ and let F be an a-arrangeable r-chromatic graph with at most ξn vertices and with maximum degree $\Delta(F) \leq \sqrt{n}/\log n$. Then G contains an F-factor.

Our second application is a universality result for random graphs $\mathcal{G}(n,p)$ with constant p (that is, a graph on vertex set $[n]$ for which every $e \in \binom{[n]}{2}$ is inserted as an edge independently with probability p). A graph G is called *universal* for a family $\mathcal{H}$ of graphs if G contains a copy of each graph in $\mathcal{H}$ as a subgraph. For instance, graphs that are universal for the family of forests, of planar graphs and of bounded degree graphs have been investigated (see [2] and the references therein).

Here we consider the class

$$\mathcal{H}_{n,a,\xi} := \{H : |H| = n,\ H \text{ is } a\text{-arrangeable},\ \Delta(H) \leq \xi n/\log n\}$$

of arrangeable graphs whose maximum degree is allowed to grow with n. Using Theorem 5, we show that with high probability $\mathcal{G}(n,p)$ contains a copy of each graph in $\mathcal{H}_{n,a,\xi}$

(see Section 6). Universality problems for bounded degree graphs in (subgraphs of) random graphs with constant p were also considered in [10]. Another result for subgraphs of potentially growing degree and p tending to 0 can be found in [27]. Theorem 2.1 of [27] implies that any a-arrangeable graph of maximum degree $o(n^{1/4})$ can be embedded into $\mathcal{G}(n,p)$ with $p > 0$ constant with high probability.

Theorem 8
For all constants a, $p > 0$ there exists $\xi > 0$ such that $\mathcal{G}(n,p)$ is universal for $\mathcal{H}_{n,a,\xi}$ with high probability.

In addition, in [5] Theorem 4 is used to establish an analogue of the Bandwidth Theorem from [4] for arrangeable graphs. A graph has *bandwidth* at most b if there is a labelling of its vertices by distinct numbers $1,\dots,n$ such that for each edge uv we have $|u-v| \le b$.

Theorem 9 (Arrangeable Bandwidth Theorem [5])
For all $r, a \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ the following holds. If H is an r-chromatic, a-arrangeable graph on n vertices with $\Delta(H) \le \sqrt{n}/\log n$ and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \ge \left(\frac{r-1}{r} + \gamma\right) n$, then there exists an embedding of H into G.

As is also shown there, this implies for example that every graph G with minimum degree at least $(\frac{3}{4} + \gamma)n$ contains *almost every* planar graph H on n vertices, provided that $\gamma > 0$. In addition it implies that almost every planar graph H has Ramsey number $R(H) \le 12|H|$.

Finally, another application of Theorem 4 appears in [1]. In that paper Allen, Skokan, and Würfl prove the following result, closing a gap left in the analysis of large planar subgraphs of dense graphs by Kühn, Osthus, and Taraz [26] and Kühn and Osthus [23].

Theorem 10 (Allen, Skokan, Würfl [1])
For every $\gamma \in (0, 1/2)$ there exists n_γ such that every graph on $n \ge n_\gamma$ vertices with minimum degree at least γn contains a planar subgraph with $2n - 4k$ edges, where k is the unique integer such that $k \le 1/(2\gamma) < k + 1$.

Methods. To prove the full version of our Arrangeable Blow-up Lemma (Theorem 4), we proceed in two steps. Firstly, we use a random greedy algorithm to embed an almost spanning subgraph H' of the target graph H into the host graph G (proving Theorem 6 along the way). Secondly, we complete the embedding by finding matchings in suitable auxiliary graphs which concern the remaining vertices in $V(H) \setminus V(H')$ and the unused vertices V^{Free} of G. The first step uses an approach similar to the one of Komlós, Sárközy, and Szemerédi [17]. The second step utilises ideas from Rödl and Ruciński's [28]. Let us briefly comment on the similarities and differences.

The use of a random greedy algorithm to prove the Blow-up Lemma appears in [17]. The idea is intuitive and simple: Order the vertices of the target graph H' arbitrarily and consecutively embed them into the host graph G, in each step choosing a random image vertex $\varphi(x)$ in the set $A(x)$ of those vertices which are still possible as images for the vertex x of H' we are currently embedding. If for some unembedded vertex x the set $A(x)$ gets too small, then call x critical and embed it immediately, but still randomly in $A(x)$. Our random greedy algorithm proceeds similarly, with one main difference. We cannot use an arbitrary order of the vertices of H', but have to use one which respects the arrangeability bound. Consequently, we also cannot embed critical vertices immediately – each vertex has to be embedded when it is its turn according to the given order. So we need a different strategy for dealing with critical vertices. We solve this problem by reserving a linear sized set of special vertices in G for the embedding of critical vertices, which are very few.

The second step is more intricate. Similarly to the approach in [28] we construct for each cluster V_i an auxiliary bipartite graph F_i with the classes $X_i \setminus V(H')$ and $V_i \cap V^{Free}$ and an edge between $x \in V(H)$ and $v \in V(G)$ whenever embedding x into v is a permissible extension of the partial embedding from the first step. Moreover, we guarantee that $V(H) \setminus V(H')$ is a stable set. Then, clearly, if each F_i has a perfect matching, there is an embedding of H into G. So the question remains how to show that the auxiliary graphs have perfect matchings. Rödl and Ruciński approach this by showing that their auxiliary graphs are super-regular. We would like to use a similar strategy, but there are two main difficulties. Firstly, because the degrees in our auxiliary graphs vary greatly, they cannot be super-regular. Hence we have to appropriately adjust this notion to our setting, which results in a property that we call weighted super-regular. Secondly, the proof that our auxiliary graphs are weighted super-regular now has to proceed quite differently, because we are dealing with the arrangeable graphs.

Structure. This paper is organised as follows. In Section 2 we provide notation and some tools. In Section 3 we show how to embed almost spanning arrangeable graphs, which will prove Theorem 6. In Section 4 we extend this to become a spanning embedding, proving Theorem 4. At the end of Section 4, we also outline how a similar argument gives Theorem 5. In Section 5 we explain why the degree bounds in the new versions of the Blow-up Lemma and the requirements for the image restrictions are essentially best possible. In Section 6, we give the proofs for our applications, Theorem 7 and Theorem 8.

2 Notation and preliminaries

All logarithms are to base e. For a graph G we write $V(G)$ for its vertex set, $E(G)$ for its edge set and denote the number of its vertices by $|G|$, its *maximum degree* by $\Delta(G)$ and its *minimum degree* by $\delta(G)$. Let $u, v \in V(G)$ and $U, W \subset V(G)$. The *neighbourhood* of u in G is denoted by $N_G(u)$, the neighbourhood of u in the set U by $N_G(u, U) := N_G(u) \cap U$. Similarly $N_G(U) = \bigcup_{x \in U} N_G(x)$ and $N_G(U, W) := N_G(U) \cap W$. The *co-degree* of u and v is $\deg_G(u, v) = |N_G(u) \cap N_G(v)|$. We often omit the subscript G.

For easier reading, we will often use x, y or z for vertices in the graph H that we are embedding, and u, v, w for vertices of the host graph G.

We shall also use the following version of the Hajnal-Szemerédi Theorem [9].

Theorem 11
Every graph G on n vertices and maximum degree $\Delta(G)$ can be partitioned into $\Delta(G)+1$ stable sets of size $\lfloor n/(\Delta(G)+1)\rfloor$ or $\lceil n/(\Delta(G)+1)\rceil$ each.

2.1 Arrangeability

Let H be a graph and $(x_1, x_2, \ldots, x_n)$ be an a-arrangeable ordering of its vertices. We write $x_i \prec x_j$ if and only if $i < j$ and say that x_i is *left* of x_j and x_j is *right* of x_i. We write $N^-(x) := \{y \in N_H(x) : y \prec x\}$ and $N^+(x) := \{y \in N_H(x) : x \prec y\}$ and call these the set of *predecessors* or the set of *successors* of x respectively. Predecessors and successors of vertex sets and in vertex sets are defined accordingly. Then $|N^+(x)| \leq \Delta(H)$ for all $x \in V(H)$ and the definition of arrangeability says that $N^-\big(N^+(x_i)\big) \cap \{x_1, \ldots, x_i\}$ is of size at most a for each $i \in [n]$. Moreover, it follows that all $x \in V(H)$ satisfy $|N^-(x)| \leq a$ and

$$e(H) = \sum_{x \in V(H)} |N^+(x)| = \sum_{x \in V(H)} |N^-(x)| \leq an\,. \tag{1}$$

In the proof of our main theorem, it will turn out to be desirable to have a vertex ordering which is not only arrangeable, but also has the property that its final μn vertices form a stable set. More precisely we require the following properties.

Definition 12 (stable ending)
Let $\mu > 0$ and let $H = (X_1 \dot\cup \ldots \dot\cup X_r, E)$ be an r-partite, a-arrangeable graph with partition classes of order $|X_i| = n_i$ with $\sum_{i\in[r]} n_i = n$. Let $(v_1, \ldots, v_n)$ be an a-arrangeable ordering of H. We say that the ordering has a stable ending of order μn *if $W = \{v_{(1-\mu)n+1}, \ldots, v_n\}$ has the following properties*

(i) $|W \cap X_i| = \mu n_i$ for every $i \in [r]$,

(ii) $H[W]$ is a stable set.

The next lemma shows that an arrangeable order of a graph can be reordered to have a stable ending while only slightly increasing the arrangeability bound.

Lemma 13
Let a, Δ_R, κ be integers and let H be an a-arrangeable graph that has a κ-balanced R-partition with $\Delta(R) < \Delta_R$. Then H has a $(5a^2\kappa\Delta_R)$-arrangeable ordering with stable ending of order μn, where $\mu = 1/(10a(\kappa\Delta_R)^2)$.

Proof. Let $X = X_1 \dot\cup \ldots \dot\cup X_r$ be a κ-balanced R-partition of H with $|X_i| = n_i$. Further let $(x_1, \ldots, x_n)$ be any a-arrangeable ordering of H. In a first step we will find a stable set $W \subseteq X$ with $|W \cap X_i| = \mu n_i$ for $\mu = 1/(10a(\kappa\Delta_R)^2)$. Note that for every $i \in [r]$ a vertex

$x \in X_i$ has only neighbours in sets X_j with $ij \in E(R)$. Further $H[X_i \cup \{X_j : ij \in E(R)\}]$ has at most $\kappa\Delta_R n_i$ vertices and is a-arrangeable. Therefore

$$\sum_{w \in X_i} \deg(w) \overset{(1)}{\leq} 2a\kappa\Delta_R n_i .$$

It follows that at least half the vertices $w \in X_i$ have $\deg(w) \leq 4a\kappa\Delta_R$. Let W_i' be the set of these vertices and m_i' be their number.

Now we greedily find a stable set $W \subseteq \bigcup_{i\in[r]} W_i'$ as follows. In the beginning we set $W = \emptyset$. Then we iteratively select an $i \in [r]$ with

$$|X_i \cap W|/n_i = \min_{j\in[r]} |X_j \cap W|/n_j \,, \tag{2}$$

choose an arbitrary vertex $x \in W_i'$, move it to W and delete x from W_i' and $N_H(x)$ from W_j' for all $j \in [r]$. We perform this operation until we have found a stable set W with $|W \cap X_i| = \mu n_i$ for all $i \in [r]$ or we attempt to choose a vertex from an empty set W_{i^*}'.

So assume that, at some point, we try to choose a vertex from an empty set W_{i^*}'. For each $i \in [r]$ let m_i be the number of vertices chosen from X_i (and moved to W) so far. Moreover, let $i \in [r]$ be such that $m_i < \mu n_i$ and consider the last step when a vertex from X_i was chosen. Before this step, $m_i - 1$ vertices of X_i and at most m_{i^*} vertices of X_{i^*} have been chosen. By (2) we thus have $(m_i - 1)/n_i \leq m_{i^*}/n_{i^*}$, which implies $m_i \leq \kappa m_{i^*} + 1$ because $n_i \leq \kappa n_{i^*}$. Hence, since W_{i^*}' became empty, we have

$$\begin{aligned} n_{i^*}/2 \leq m_{i^*}' &\leq m_{i^*} + \sum_{\{i^*,i\}\in E(R)} m_i 4a\kappa\Delta_R \\ &\leq m_{i^*} + (\Delta_R - 1)(\kappa m_{i^*} + 1)4a\kappa\Delta_R \leq m_{i^*}\, 5a(\kappa\Delta_R)^2 \,. \end{aligned}$$

Thus $m_{i^*} \geq n_{i^*}/(10a(\kappa\Delta_R)^2)$. Since we then try to choose from W_{i^*}' we must have $m_{i^*}/n_{i^*} \leq m_i/n_i$ by (2), which implies $m_i \geq n_i/(10a(\kappa\Delta_R)^2) = \mu n_i$. Hence we indeed find a stable set W with $|W \cap X_i| = \mu n_i$ for all $i \in [r]$.

Given this stable set W we define a new ordering in which these vertices are moved to the end in order to form the stable ending. To make this more precise let $(x_1', \dots, x_n')$ be the vertex ordering obtained from $(x_1, \dots, x_n)$ by moving all vertices of W to the end (in any order). It remains to prove that $(x_1', \dots, x_n')$ is $(5a^2\kappa\Delta_R)$-arrangeable. Let $L_i' = \{x_1', \dots, x_i'\}$ be the vertices left of x_i' including x_i' and $R_i' = \{x_{i+1}', \dots, x_n'\}$ be the vertices right of x_i' in the new ordering. We have to show that

$$\big|N\big(N(x_i', R_i'), L_i'\big)\big| \leq 5a^2\kappa\Delta_R$$

for all $i \in [n]$. This is obvious for the vertices in W because they are now at the end and W is stable. For $x_i \notin W$ let $N_i' = N\big(N(x_i, R_i'), L_i'\big)$ be the set of predecessors of successors of x_i in the new ordering. N_i is defined analogously for the original ordering. Then all vertices in $N_i' \setminus N_i$ are neighbours of predecessors y of x_i in the original ordering with $y \in W$. There are at most a such left-neighbours of x_i and each of these has at most $4a\kappa\Delta_R$ neighbours by definition of W. Hence

$$|N_i'| \leq |N_i| + a \cdot 4a\kappa\Delta_R \leq a + 4a^2\kappa\Delta_R \leq 5a^2\kappa\Delta_R \,. \qquad \square$$

2.2 Weighted regularity

In our proof we shall make use of a weighted version of ε-regularity. More precisely, we will have to deal with a bipartite graph whose vertices have very different degrees. The idea is then to give each vertex a weight antiproportional to its degree and then say that the graph is weighted regular if the following holds.

Definition 14 (Weighted regular pairs)
Let $\varepsilon > 0$ and consider a bipartite graph $G = (A \mathbin{\dot\cup} B, E)$ with a weight function $\omega : A \to [0, 1]$. For $A' \subseteq A$, $B' \subseteq B$ we define the weighted density

$$d_\omega(A', B') := \frac{\sum_{x \in A'} \omega(x)|N(x, B')|}{|A'| \cdot |B'|} .$$

We say that the pair (A, B) with weight function ω is weighted ε-regular *(with respect to ω) if for any $A' \subseteq A$ with $|A'| \geq \varepsilon|A|$ and any $B' \subseteq B$ with $|B'| \geq \varepsilon|B|$ we have*

$$|d_\omega(A, B) - d_\omega(A', B')| \leq \varepsilon .$$

Many results for ε-regular pairs carry over to weighted ε-regular pairs. For one, subpairs of weighted regular pairs are weighted regular.

Proposition 15
Let $G = (A \mathbin{\dot\cup} B, E)$ with weight function $\omega : A \to [0, 1]$ be weighted ε-regular. Further let $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$ for some $\gamma \geq \varepsilon$ and set $\varepsilon' := \max\{2\varepsilon, \varepsilon/\gamma\}$. Then $(A' \mathbin{\dot\cup} B', E \cap A' \times B')$ is a weighted ε'-regular pair with respect to the restricted weight function $\omega' : A' \to [0, 1]$, $\omega'(x) = \omega(x)$.

Proof. Let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \gamma|A|$, $|B'| \geq \gamma|B|$ be arbitrary. The definition of weighted ε-regularity implies that $|d_\omega(A, B) - d_\omega(A', B')| \leq \varepsilon$. Moreover, $|d_\omega(A, B) - d_\omega(A^*, B^*)| \leq \varepsilon$ for all $A^* \subseteq A'$ and $B^* \subseteq B'$ with $|A^*| \geq (\varepsilon/\gamma)|A'| \geq \varepsilon|A|$, $|B^*| \geq (\varepsilon/\gamma)|B'| \geq \varepsilon|B|$ for the same reason. It follows by triangle inequality that $|d_\omega(A', B') - d_\omega(A^*, B^*)| \leq 2\varepsilon$. Hence $(A' \mathbin{\dot\cup} B', E \cap A' \times B')$ with weight function $\omega' : A' \to [0, 1]$ is a weighted ε'-regular pair where $\varepsilon' = \max\{2\varepsilon, \varepsilon/\gamma\}$. $\square$

If most vertices of a bipartite graph have the 'right' degree and most pairs have the 'right' co-degree then the graph is an ε-regular pair. This remains to be true for weighted regular pairs and weighted degrees and co-degrees.

Definition 16 (Weighted degree and co-degree)
Let $G = (A \mathbin{\dot\cup} B, E)$ be a bipartite graph and $\omega : A \to [0, 1]$. For $x, y \in A$ we define the weighted degree *of x as $\deg_\omega(x) := \omega(x)|N(x, B)|$ and the* weighted co-degree *of x and y as $\deg_\omega(x, y) := \omega(x)\omega(y)|N(x, B) \cap N(y, B)|$.*

A proof of the following lemma can be found in the Appendix.

Lemma 17
Let $\varepsilon > 0$ and $n \geq \varepsilon^{-6}$. Further let $G = (A \mathbin{\dot\cup} B, E)$ be a bipartite graph with $|A| = |B| = n$ and let $\omega : A \to [\varepsilon, 1]$ be a weight function for G. If

(i) $|\{x \in A : |\deg_\omega(x) - d_\omega(A,B)n| > \varepsilon^{14}n\}| < \varepsilon^{12}n$ *and*
(ii) $|\{\{x,y\} \in \binom{A}{2} : |\deg_\omega(x,y) - d_\omega(A,B)^2 n| \geq \varepsilon^9 n\}| \leq \varepsilon^6\binom{n}{2}$
then (A,B) *is a weighted* 3ε*-regular pair.*

It is well known that a balanced (ε,δ)-super-regular pair has a perfect matching if $\delta > 2\varepsilon$ (see, e.g., [28]). Similarly, balanced weighted regular pairs with an appropriate minimum degree bound have perfect matchings (see the Appendix for a proof).

Lemma 18
Let $\varepsilon > 0$ *and let* $G = (A \,\dot\cup\, B, E)$ *with* $|A| = |B| = n$ *and weight function* $\omega : A \to [\sqrt{\varepsilon}, 1]$ *be a weighted* ε*-regular pair. If* $\deg(x) > 2\sqrt{\varepsilon}n$ *for all* $x \in A \cup B$ *then* G *contains a perfect matching.*

2.3 Chernoff type bounds

Our proofs will heavily rely on the probabilistic method. In particular we will want to bound random variables that are close to being binomial. By close to we mean that the individual events are not necessarily independent but occur with certain probability even if condition on the outcome of other events. The following two variations on the classical bound by Chernoff make this more precise.

Lemma 19
Let $0 \leq p_1 \leq p_2 \leq 1$, $0 < c \leq 1$. *Further let* $\mathcal{A}_i$ *for* $i \in [n]$ *be 0-1-random variables and set* $\mathcal{A} := \sum_{i\in[n]} \mathcal{A}_i$. *If*

$$p_1 \leq \mathbb{P}\left[\mathcal{A}_i = 1 \;\middle|\; \begin{array}{l} \mathcal{A}_j = 1 \textit{ for all } j \in J \textit{ and} \\ \mathcal{A}_j = 0 \textit{ for all } j \in [i-1]\setminus J \end{array}\right] \leq p_2$$

for every $i \in [n]$ *and every* $J \subseteq [i-1]$ *then*

$$\mathbb{P}[\mathcal{A} \leq (1-c)p_1 n] \leq \exp\left(-\frac{c^2}{3}p_1 n\right)$$

and

$$\mathbb{P}[\mathcal{A} \geq (1+c)p_2 n] \leq \exp\left(-\frac{c^2}{3}p_2 n\right).$$

Similarly we can state a bound on the number of tuples of certain random variables.

Lemma 20
Let $0 < p$ *and* $a, m, n \in \mathbb{N}$. *Further let* $\mathcal{I} \subseteq \mathcal{P}([n]) \setminus \{\emptyset\}$ *be a collection of* m *disjoint sets with at most* a *elements each. For every* $i \in [n]$ *let* $\mathcal{A}_i$ *be a 0-1-random variable. Further assume that for every* $I \in \mathcal{I}$ *and every* $k \in I$ *we have*

$$\mathbb{P}\left[\mathcal{A}_k = 1 \;\middle|\; \begin{array}{l} \mathcal{A}_j = 1 \textit{ for all } j \in J \\ \mathcal{A}_j = 0 \textit{ for all } j \in [k-1]\setminus J \end{array}\right] \geq p$$

for every $J \subseteq [k-1]$ *with* $[k-1] \cap I \subseteq J$. *Then*

$$\mathbb{P}\left[\left|\{I \in \mathcal{I} : \mathcal{A}_i = 1 \textit{ for all } i \in I\}\right| \geq \tfrac{1}{2}p^a m\right] \geq 1 - 2\exp\left(-\frac{1}{12}p^a m\right).$$

The proofs for both lemmas can be found in the Appendix. The first one is very close to the proof of the classical Chernoff bound while the second proof builds on the fact that the events $[\mathcal{A}_i = 1 \text{ for all } i \in I]$ have probability at least p^a for every $I \in \mathcal{I}$. In particular, in the special case $a = 1$, Lemma 19 implies Lemma 20.

3 An almost spanning version of the Blow-up Lemma

This section is dedicated to the proof of Theorem 6 which is a first step towards Theorem 4. We give a randomised algorithm for the embedding of an almost spanning subgraph H' into G and show that it is well defined and that it succeeds with positive probability.

This embedding of H' is later extended to the embedding of a spanning subgraph H in Section 4. Applying the randomised algorithm to H while only embedding H' provides the structural information necessary for the extension of the embedding. It is for this reason that we define a graph H while only embedding a subgraph $H' \subseteq H$ into G in this section.

Remark *In the following we shall always assume that each super-regular pair* (V_i, V_j) *appearing in the proof has density*

$$d(V_i, V_j) = \delta$$

exactly*, and minimum degree*

$$\min_{v \in V_i} \deg(v, V_j) \geq \tfrac{1}{2}\delta|V_j|\,, \quad \min_{v \in V_j} \deg(v, V_i) \geq \tfrac{1}{2}\delta|V_i|\,, \tag{3}$$

since otherwise we can simply appropriately delete random edges to obtain this situation (while possibly increasing regularity to 2ε*).*

3.1 Constants, constants

Since there will be plenty of constants involved in the following proofs we give a short overview first.

Δ_R: the maximum degree of R is *strictly* smaller than Δ_R
r: the number of clusters
a: the arrangeability of H
s: the chromatic number of H
δ: the density of the pairs (V_i, V_j) in G
μ: the proportion of G that will be left after embedding H
ξ: some constant in the degree-bound of H
ε: the regularity of the pairs (V_i, V_j) in G
ε': the weighted regularity of the auxiliary graphs $F_i(t)$
κ: the maximum quotient between cluster sizes
γ: a threshold for moving a vertex into the critical set

λ: the fraction of vertices whose predecessors receive a special embedding
α: the fraction of vertices with image restrictions
c: the relative size of the image restrictions
C: the maximum number of image restrictions per cluster

Now let $C, a, \Delta_R, \kappa \in \mathbb{N}$ and $\delta, c, \mu > 0$ be given. We define the following constants.

$$\gamma = \frac{c}{2}\frac{\mu}{10}\delta^a , \tag{4}$$

$$\lambda = \frac{1}{25a}\delta\gamma , \tag{5}$$

$$\varepsilon' = \min\left\{\left(\frac{\lambda\delta^a}{6 \cdot 2^{a^2+1}3^a}\right)^2, \left(\frac{7\gamma}{30}\right)^2\right\} , \tag{6}$$

$$\varepsilon = \min\left\{\frac{1}{\Delta_R(1+C)2^{a+1}}\varepsilon', \left(\frac{\varepsilon'}{3}\right)^{36}\right\} , \tag{7}$$

$$\alpha = \frac{\sqrt{\varepsilon}}{6} . \tag{8}$$

Furthermore, let r be given. Then we choose

$$\xi = \frac{8\varepsilon^2}{9\gamma^2\kappa r} . \tag{9}$$

Moreover, we ensure that n_0 is big enough to guarantee

$$\sqrt{n_0} \geq 48\frac{3^a 2^{a^2+1} a\kappa r}{\lambda\delta^a} , \quad n_0 \geq 60\frac{\kappa r}{\varepsilon^2\delta\mu}\log(12(n_0)^2), \text{ and } \quad \log n_0 \geq 36\frac{2^{a^2}a^2\kappa r}{\lambda} . \tag{10}$$

All logarithms are base e. In short, the constants used relate as

$$0 < \xi \ll \varepsilon \ll \alpha \ll \varepsilon' \ll \lambda \ll \gamma \ll \mu, \delta \leq 1 .$$

Moreover, $\varepsilon \ll 1/\Delta_R$. Note that it follows from these definitions that $(1+\varepsilon/\delta)^a \leq 1+\sqrt{\varepsilon}/3$ and $(1-\varepsilon/\delta)^a \geq 1-\sqrt{\varepsilon}/3$ which implies

$$\frac{(\delta+\varepsilon)^a}{1+\sqrt{\varepsilon}/3} \leq \delta^a \leq \frac{(\delta-\varepsilon)^a}{1-\sqrt{\varepsilon}/3}, \quad \text{in particular} \quad (\delta-\varepsilon)^a \geq \frac{9}{10}\delta^a. \tag{11}$$

3.2 The randomised greedy algorithm

Let $V(H) = (x_1, \ldots, x_n)$ be an a-arrangeable ordering of H and let $H' \subseteq H$ be a subgraph induced by $\{x_1, \ldots, x_{(1-\mu)n}\}$. In this section we define a *randomised greedy algorithm* (RGA) for the embedding of $V(H')$ into $V(G)$. This algorithm processes the vertices of H vertex by vertex and thereby defines an embedding φ of H' into G. We say that vertex x_t gets embedded in time step t where t runs from 1 to $T = |H'|$. Accordingly $t(x) \in [n]$ is defined to be the time step in which vertex x will be embedded.

We explain the main ideas before giving an exact definition of the algorithm.

Preparing H: Recall that S_i is the set of *image restricted vertices* in X_i and set $S := \bigcup S_i$. We define L_i^* to be the last λn_i vertices in $X_i \setminus N(S)$ in the arrangeable ordering. Moreover, we define $X_i^* := N^-(L_i^*) \cup S_i$ and $X^* := \bigcup X_i^*$. Those vertices will be called the *important vertices.* The name indicates that they will play a major rôle for the spanning embedding. Important vertices shall be treated specially by the embedding algorithm. The a-arrangeability of H implies that

$$|X_i^*| \leq a\lambda n_i + \alpha n_i \tag{12}$$

for all $i \in [r]$.

Preparing G: Before we start embedding into G we randomly set aside $(\mu/10)n_i$ vertices in V_i for each $i \in [r]$. We denote these sets by V_i^{s} and call them the *special vertices.* All remaining vertices, i.e., $V_i^{\mathrm{o}} := V_i \setminus V_i^{\mathrm{s}}$ will be called *ordinary vertices.* As the name suggests our algorithm will try to embed most vertices of H' into the sets V_i^{o} and only if this fails resort to embedding into V_i^{s}. The idea is that the special vertices will be reserved for the important vertices and for those vertices in H' whose embedding turns out to be intricate. We define

$$V^{\mathrm{o}} := \bigcup_{i=1}^{r} V_i^{\mathrm{o}}, \quad V^{\mathrm{s}} := \bigcup_{i=1}^{r} V_i^{\mathrm{s}}.$$

Note that $V^{\mathrm{o}} \,\dot\cup\, V^{\mathrm{s}}$ defines a partition of $V(G)$.

Candidate sets: While our embedding process is running, more and more vertices of G will be used up to accommodate vertices of H. For each time step $t \in [n]$ we denote by $V^{\mathit{Free}}(t) := V(G) \setminus \{v \in V(G) : \exists t' < t : \varphi(x_{t'}) = v\}$ the set of vertices where no vertex of H has been embedded yet. Obviously $\varphi(x_t) \in V^{\mathit{Free}}(t)$ for all t.

The algorithm will define sets $C_{t,x} \subseteq V(G)$ for $1 \leq t \leq T$, $x \in V(H)$, which we will call the *candidate set* for x at time t. Analogously

$$A_{t,x} := C_{t,x} \cap V^{\mathit{Free}}(t)$$

will be called the *available candidate set* for x at time t. Again we distinguish between the *ordinary candidate* set $C_{t,x}^{\mathrm{o}} := C_{t,x} \cap V^{\mathrm{o}}$ and the *special candidate set* $C_{t,x}^{\mathrm{s}} := C_{t,x} \cap V^{\mathrm{s}}$ or their respective available version $A_{t,x}^{\mathrm{o}} := A_{t,x} \cap V^{\mathrm{o}}$ and $A_{t,x}^{\mathrm{s}} := A_{t,x} \cap V^{\mathrm{s}}$.

Finally we define a set $Q(t) \subseteq V(H)$ and call it the *critical set* at time t. $Q(t)$ will contain the vertices whose available candidate set got too small at time t or earlier.

Algorithm RGA

Initialisation

Randomly select $V_i^{\mathrm{s}} \subseteq V_i$ with $|V_i^{\mathrm{s}}| = (\mu/10)|V_i|$ for each $i \in [r]$. For $x \in X_i \setminus S_i$ set $C_{1,x} = V_i$ and for $x \in S_i$ set $C_{1,x} = I(x)$. Set $Q(1) = \emptyset$.

Check that for every $i \in [r]$, $v \in V_i$, and every $j \in N_R(i)$ we have

$$\left| \frac{|N_G(v) \cap V_j^{\mathrm{s}}|}{|V_j^{\mathrm{s}}|} - \frac{|N_G(v) \cap V_j|}{|V_j|} \right| \leq \varepsilon\,. \tag{13}$$

Further check that every $x \in S_i$ has

$$|C_{1,x}^{\mathrm{s}}| = |I(x) \cap V_i^{\mathrm{s}}| \geq \tfrac{1}{20} c\mu\, n_i\,. \tag{14}$$

Halt with failure if any of these does not hold.

EMBEDDING STAGE

For $t \geq 1$, **repeat** the following steps.

Step 1 – Embedding x_t: Let $x = x_t$ be the vertex of H to be embedded at time t. Let $A'_{t,x}$ be the set of vertices $v \in A_{t,x}$ which satisfy (15) and (16) for all $y \in N^+(x)$:

$$(\delta - \varepsilon)|C_{t,y}^{\mathrm{o}}| \leq |N_G(v) \cap C_{t,y}^{\mathrm{o}}| \leq (\delta + \varepsilon)|C_{t,y}^{\mathrm{o}}|, \tag{15}$$
$$(\delta - \varepsilon)|C_{t,y}^{\mathrm{s}}| \leq |N_G(v) \cap C_{t,y}^{\mathrm{s}}| \leq (\delta + \varepsilon)|C_{t,y}^{\mathrm{s}}|. \tag{16}$$

Choose $\varphi(x)$ uniformly at random from

$$A(x) := \begin{cases} A_{t,x}^{\mathrm{o}} \cap A'_{t,x} & \text{if } x \notin X^* \text{ and } x \notin Q(t), \\ A_{t,x}^{\mathrm{s}} \cap A'_{t,x} & \text{else.} \end{cases} \tag{17}$$

Step 2 – Updating candidate sets: for each unembedded vertex $y \in V(H)$, set

$$C_{t+1,y} := \begin{cases} C_{t,y} \cap N_G(\varphi(x)) & \text{if } y \in N^+(x), \\ C_{t,y} & \text{otherwise.} \end{cases}$$

Step 3 – Updating critical vertices: We will call a vertex $y \in X_i$ *critical* if $y \notin X_i^*$ and

$$|A_{t+1,y}^{\mathrm{o}}| < \gamma n_i. \tag{18}$$

Obtain $Q(t+1)$ by adding to $Q(t)$ all critical vertices that have not been embedded yet. Set $Q_i(t+1) = Q(t+1) \cap X_i$.

Halt with failure if there is $i \in [r]$ with

$$|Q_i(t+1)| > \varepsilon' n_i\,. \tag{19}$$

Else, if there are no more unembedded vertices left in $V(H')$ **halt with success**, **otherwise** set $t \leftarrow t+1$ and go back to *Step 1*.

We have now defined our randomised greedy algorithm for the embedding of an almost spanning subgraph H' into G. The rest of this section is to prove that it succeeds with positive probability. This then implies Theorem 6.

In order to analyse the RGA we define auxiliary graphs which describe possible embeddings of vertices of H' into G. These auxiliary graphs inherit some kind of regularity from G with positive probability. We show that the algorithm terminates successfully whenever this happens.

In the subsequent Section 3.3 we show that conditions (13) and (14) hold with probability at least 5/6. The INITIALISATION of the RGA succeeds whenever this happens. Moreover, we prove that the embedding of each vertex is randomly chosen from a set of linear size in Step 1 of the EMBEDDING STAGE.

In Section 3.4 we define auxiliary graphs and derive that all auxiliary graphs are weighted regular with probability at least 5/6. We also show that condition (19) never holds if this is the case. Thus the EMBEDDING STAGE also terminates successfully with probability at least 5/6.

We conclude that the whole RGA succeeds with probability at least 2/3. This implies Theorem 6.

3.3 Initialisation and Step 1

This section is to prove that the INITIALISATION of the RGA succeeds with probability at least 5/6 and that Step 1 of the EMBEDDING STAGE always chooses vertices from a non-empty set.

Lemma 21
The INITIALISATION *succeeds with probability at least* 5/6*, i.e. both condition* (13) *and* (14) *hold for every* $i \in [r]$*,* $v \in V_i$*,* $j \in [r] \setminus \{i\}$*, and* $x \in S_i$ *with probability* 5/6.

Proof of Lemma 21. Fix one $v \in V_i$, $j \in [r] \setminus \{i\}$. Since V_j^{s} is a randomly chosen subset of V_j we have

$$\mathbb{E}[|N_G(v) \cap V_j^{\mathrm{s}}|] = |N_G(v) \cap V_j| \frac{|V_j^{\mathrm{s}}|}{|V_j|} \geq \frac{\delta}{2} n_j \frac{\mu}{10} .$$

It follows from a Chernoff bound (see Theorem 41 in the Appendix) that

$$\mathbb{P}\left[\left| |N_G(v) \cap V_j^{\mathrm{s}}| - |N_G(v) \cap V_j| \frac{|V_j^{\mathrm{s}}|}{|V_j|} \right| > \varepsilon |V_j^{\mathrm{s}}| \right] \leq \exp\left(-\frac{\varepsilon^2}{6} \delta n_i \frac{\mu}{10} \right) \overset{(10)}{\leq} \frac{1}{12n^2} .$$

Similarly $c|V_i^{\mathrm{s}}| \geq c\frac{\mu}{10} n_i$ and

$$\mathbb{P}[c|V_i^{\mathrm{s}}| - |I(x) \cap V_i^{\mathrm{s}}| \geq \frac{c}{2} |V_i^{\mathrm{s}}|] \leq \exp\left(-\frac{1}{12} c \frac{\mu}{10} n_i \right) \leq \frac{1}{12n} .$$

A union bound over all $i \in [r]$, $v \in V_i$ and $j \in N_R(i)$ or over all $x \in S_i$ finishes the proof. $\square$

Let us write $\pi(t, x)$ for the number of predecessors of x that already got embedded by time t:

$$\pi(t, x) := |\{t' < t : \{x, x_{t'}\} \in E(H)\}|.$$

Obviously $\pi(t, x) \leq a$ by the definition of arrangeability.

Lemma 22
Let $x \in X_i \setminus S_i$ and $t \le T$ be arbitrary. Then

$$(1-\mu/10)(\delta-\varepsilon)^{\pi(t,x)} n_i \le |C^{\mathrm{o}}_{t,x}| \le (1-\mu/10)(\delta+\varepsilon)^{\pi(t,x)} n_i \,,$$
$$(\mu/10)(\delta-\varepsilon)^{\pi(t,x)} n_i \le |C^{\mathrm{s}}_{t,x}| \le (\mu/10)(\delta+\varepsilon)^{\pi(t,x)} n_i \,.$$

If $x \in S_i$, $t \le T$ then

$$\frac{9}{10}\gamma n_i \le |C^{\mathrm{s}}_{t,x}| \,.$$

Proof. The Initialisation of the RGA defines the candidate sets such that $|C^{\mathrm{o}}_{1,x}| = (1-\mu/10)n_i$ and $|C^{\mathrm{s}}_{1,x}| = (\mu/10)n_i$ for every $x \in X_i \setminus S_i$. In the Embedding Stage conditions (15) and (16) guarantee that $C^{\mathrm{o}}_{t,x}$ and $C^{\mathrm{s}}_{t,x}$ respectively shrink by a factor of $(\delta \pm \varepsilon)$ whenever a vertex in $N^-(x)$ is embedded.

If $x \in S_i$ we still have $|C^{\mathrm{s}}_{1,x}| \ge (c\mu/20)n_i$ by (14). The statement follows as conditions (15) and (16) again guarantee that $C^{\mathrm{s}}_{t,x}$ shrinks at most by a factor of $(\delta-\varepsilon)^a$. Moreover, $\frac{1}{20}c\mu(\delta-\varepsilon)^a \ge \frac{9}{10}\gamma$ by (11) and the definition of γ. □

We now argue that $\varphi(x)$ is chosen from a non-empty set at the end of Step 1 in the Embedding Stage. In fact, we will show that $\varphi(x)$ is chosen from a set of size linear in n_i.

Lemma 23
For any vertex $x \in X_i$ that gets embedded in the Embedding Stage *$\varphi(x)$ is chosen randomly from a set $A(x)$ of size at least $(\gamma/2)n_i$.*
Moreover, if x gets embedded into V^{s}_i

$$|X^*_i| + |Q_i(t(x))| + |A^{\mathrm{s}}_{t(x),x} \setminus A(x)| \le \frac{\delta}{18}|C^{\mathrm{s}}_{t(x),x}| \,.$$

If the RGA completes the Embedding Stage *successfully but $x \in X_i$ does not get embedded in the* Embedding Stage *we have*

$$|A^{\mathrm{s}}_{T,x}| \ge \frac{7\gamma}{10} n_i \,.$$

Proof. We claim that any $x \in X_i$ that gets embedded into V^σ_i with $\sigma \in \{\mathrm{o},\mathrm{s}\}$ during the Embedding Stage has

$$|A^\sigma_{t(x),x}| \ge \frac{7\gamma}{10} n_i \,. \tag{20}$$

We will establish equation (20) at the end of this proof.

In order to show the first statement of the lemma we now bound $|A^\sigma_{t(x),x} \setminus A(x)|$ for any $x \in X_i$ that gets embedded into V^σ_i with $\sigma \in \{\mathrm{o},\mathrm{s}\}$ during the Embedding Stage, i.e., we determine the number of vertices that potentially violate conditions (15) or (16). As H is a-arrangeable, the vertices $y \in N^+(x)$ share at most 2^a distinct ordinary candidate sets $C^{\mathrm{o}}_{t(x),y}$ in each V_j. The number of special candidate sets $C^{\mathrm{s}}_{t(x),y}$ in each V_j might be larger by a factor of C as they arise from the intersection with at most C sets $I_{j,k}$ (with $k \in [C]$) which are the image restrictions. Moreover, there are less than Δ_R many sets V_j with $j \in N_R(i)$ bounding the total number of candidate sets we have to care for by $\Delta_R(1+C)2^a$.

As we embed x into an ε-regular pair there are at most $2\varepsilon n_i$ vertices $v \in A^{\sigma}_{t(x),x}$ for each $C^{\mathrm{o}}_{t(x),y}$ that violate (15) (and the same number for each $C^{\mathrm{s}}_{t(x),y}$ that violate (16)) with $y \in N^+(x)$. Hence

$$|A^{\sigma}_{t(x),x} \setminus A(x)| \le \Delta_R(1+C)2^{a+1}\varepsilon n_i \tag{21}$$

if x gets embedded into V_i^{σ}. Now $\Delta_R(1+C)2^{a+1}\varepsilon n_i \le \gamma/5 n_i$ by (7) and

$$|A(x)| = |A^{\sigma}_{t(x),x}| - |A^{\sigma}_{t(x),x} \setminus A(x)| \ge (\gamma/2)n_i$$

follows.

Next we show the second statement of the lemma. If $x \in X_i$ gets embedded into V_i^{s} in the Embedding Stage we conclude

$$\begin{aligned}
|X_i^*| + |Q_i(t(x))| + |A^{\mathrm{s}}_{t(x),x} \setminus A(x)| &\le (a\lambda + \alpha)\, n_i + \varepsilon' n_i + \Delta_R(1+C)2^{a+1}\varepsilon n_i \\
&\overset{(5),(7)}{\le} \left(\tfrac{1}{25}\delta\gamma + \alpha + 2\varepsilon'\right) n_i \overset{(8)}{\le} \tfrac{1}{20}\delta\gamma\, n_i \\
&\le \tfrac{\delta}{18}|C^{\mathrm{s}}_{t(x),x}|
\end{aligned}$$

where the first inequality is due to (12), (19), and (21) and the last inequality is due to $|C^{\mathrm{s}}_{t(x),x}| \ge \frac{9}{10}\gamma n_i$ by Lemma 22.

We now return to Equation (20). In order to prove it we distinguish between the two cases of (17) in *Step 1* of the Embedding Stage. If $x \notin X^*$ has never entered the critical set, it is embedded into $A^{\mathrm{o}}_{t(x),x}$ and $|A^{\mathrm{o}}_{t(x),x}| \ge (7\gamma/10)n_i$ holds by condition (18). Else x gets embedded into $A^{\mathrm{s}}_{t(x),x}$. As only vertices from $Q_i(t(x))$ or X_i^* have been embedded into V_i^{s} so far, we can bound $|A^{\mathrm{s}}_{t(x),x}|$ by

$$\begin{aligned}
|A^{\mathrm{s}}_{t(x),x}| &\ge |C^{\mathrm{s}}_{t(x),x}| - |Q_i(t(x))| - |X_i^*| \\
&\overset{(12)}{\ge} \frac{9\gamma}{10}n_i - \varepsilon' n_i - (a\lambda+\alpha)n_i \ge \frac{7\gamma}{10}n_i
\end{aligned}$$

where the second inequality is due to Lemma 22 and the third inequality is due to our choice of constants. In any case we have $|A^{\sigma}_{t(x),x}| \ge \frac{7\gamma}{10}n_i$ if x gets embedded into V_i^{σ} (with $\sigma \in \{\mathrm{o},\mathrm{s}\}$) in Step 1 of the Embedding Stage.

If the RGA completes the Embedding Stage successfully but $x \in X_i$ does not get embedded during the Embedding Stage the analogous argument gives

$$|A^{\mathrm{s}}_{T,x}| \ge |C^{\mathrm{s}}_{T,x}| - |Q_i(T)| - |X_i^*| \ge \frac{7\gamma}{10}n_i\,. \qquad \square$$

3.4 The auxiliary graph

We run the RGA as described above. In order to analyse it, we define *auxiliary graphs* $F_i(t)$ which monitor at every time step t whether a vertex $v \in V(G)$ is still contained in the candidate set of a vertex $x \in V(H)$. Let $F_i(t) := (X_i \mathbin{\dot\cup} V_i, E(F_i(t)))$ where $xv \in E(F_i(t))$ if and only if $v \in C_{t,x}$. We stress that we use the candidate sets $C_{t,x}$ and not the set of available candidates $A_{t,x}$. This is well defined as $C_{t,x} \subseteq V_i$ for every $x \in X_i$ and every t. Note that $F_i(t)$ is a balanced bipartite graph. By Lemma 22 we have

$$(\delta - \varepsilon)^{\pi(t,x)} n_i \leq \deg_{F_i(t)}(x) \leq (\delta + \varepsilon)^{\pi(t,x)} n_i \tag{22}$$

for every $x \in X_i \setminus S_i$, i.e., the degree of x in $F_i(t)$ strongly depends on the number $\pi(t,x)$ of embedded predecessors. The main goal of this section is proving, however, that if we take this into account and weight the auxiliary graphs accordingly, then they are with high probability weighted regular (see Lemma 24). It will turn out that the RGA succeeds if this is the case (see Lemma 26).

More precisely, for $F_i(t)$ we shall use the weight function $\omega_t \colon X_i \to [0,1]$ with

$$\omega_t(x) := \delta^{a-\pi(t,x)}\,. \tag{23}$$

Observe that the weight function depends on t. For nicer notation, we write $\deg_{\omega,t}(x) := \deg_{\omega(t)}(x) = \omega_t(x)|N_{F_i(t)}(x)|$ for $x \in X_i$ and $d_{\omega,t}(X,Y) := d_{\omega(t)}(X,Y)$ for $X \subseteq X_i$ and $Y \subseteq V_i$. By (22) we have

$$\deg_{\omega,t}(x) \geq \delta^{a-\pi(t,x)}(\delta - \varepsilon)^{\pi(t,x)} n_i \overset{(11)}{\geq} (1 - \sqrt{\varepsilon}/3)\delta^a n_i\,, \tag{24}$$

$$\deg_{\omega,t}(x) \leq \delta^{a-\pi(t,x')}(\delta + \varepsilon)^{\pi(t,x')} n_i \overset{(11)}{\leq} (1 + \sqrt{\varepsilon}/3)\delta^a n_i \tag{25}$$

for every $x \in X_i \setminus S_i$ and t. Thus for every $i \in [r]$ and $t \leq T$ the auxiliary graph $F_i(t)$ satisfies

$$(1 - \sqrt{\varepsilon}/2)\delta^a \overset{(8)}{\leq} (1-\alpha)(1 - \sqrt{\varepsilon}/3)\delta^a \leq d_{\omega,t}(X_i, V_i) \leq (1 + \sqrt{\varepsilon}/2)\delta^a\,. \tag{26}$$

Let $\mathcal{R}_i(t)$ denote the event that $F_i(t)$ is weighted ε'-regular for ε' as in (6). Further let $\mathcal{R}_i$ be the event that $\mathcal{R}_i(t)$ for all $t \leq T$.

Lemma 24
We run the RGA in the setting of Theorem 6. Then $\mathcal{R}_i$ holds for all $i \in [r]$ with probability at least $5/6$.

We will use Lemma 17 and weighted degrees and co-degrees to prove Lemma 24.

Proof of Lemma 24. This proof checks the conditions of Lemma 17. Let

$$W_i^{(1)}(t) = \left\{ x \in X_i : |\deg_{\omega,t}(x) - d_{\omega,t}(X_i,V_i)n_i| > \sqrt{\varepsilon} n_i \right\},$$

$$W_i^{(2)}(t) = \left\{ \{x,y\} \in \binom{X_i}{2} : |\deg_{\omega,t}(x,y) - d_{\omega,t}(X_i,V_i)^2 n_i| \geq \sqrt[4]{\varepsilon} n_i \right\}$$

be the set of vertices and pairs which deviate from the expected (co-)degree. Let $W_i^{(1)} := \bigcup_{t\in[T]} W_i^{(1)}(t)$ and $W_i^{(2)} := \bigcup_{t\in[T]} W_i^{(2)}(t)$. We have $\varepsilon' \geq 3\varepsilon^{1/36}$ by (7), and by Lemma 17 all auxiliary graphs $F_i(t)$ with $t = 1, \dots, T$ are weighted ε'-regular if both

$$|W_i^{(1)}| < \sqrt{\varepsilon} n_i \,, \tag{27}$$

$$|W_i^{(2)}| \leq \sqrt[4]{\varepsilon} \binom{n_i}{2} \,. \tag{28}$$

Thus $\mathcal{R}_i$ occurs whenever equations (27) and (28) are satisfied. We will prove that this happens for a fixed $i \in [r]$ with probability at least $1 - n_i^{-1}$, which together with a union bound over $i \in [r]$ implies the statement of the lemma.

So fix $i \in [r]$. From (24), (25) and (26) we deduce that

$$|\deg_{\omega,t}(x) - d_{\omega,t}(X_i, V_i) n_i| \leq \sqrt{\varepsilon} n_i$$

for all $x \in X_i \setminus S_i$ and every $t \leq T$. But $|S_i| \leq \alpha n_i < \sqrt{\varepsilon} n_i$ by (8) and equation (27) is thus *always* satisfied.

It remains to consider (28). To this end let P_i be the set of all pairs $\{y, z\} \in \binom{X_i \setminus S_i}{2}$ with $N^-(y) \cap N^-(z) = \emptyset$. Observe that $|\binom{X_i}{2} \setminus \binom{X_i \setminus S_i}{2}| \leq \alpha n_i^2 \leq \frac{\sqrt{\varepsilon}}{6} n_i^2$ by (8) and

$$\begin{aligned}\left|\left\{\{y,z\} \in \binom{X_i}{2} : N^-(y) \cap N^-(z) \neq \emptyset\right\}\right| &\leq a\Delta(H) n_i \leq a \frac{\xi n}{\log n} n_i \\ &\leq \frac{2a\xi\kappa r}{\log n}\binom{n_i}{2} \overset{(9)}{\leq} \sqrt{\varepsilon}\binom{n_i}{2} \,.\end{aligned}$$

Hence it suffices to show that

$$\mathbb{P}\left[|W_i^{(2)} \cap P_i| \leq \tfrac{1}{2}\sqrt[4]{\varepsilon}\binom{n_i}{2}\right] \geq \mathbb{P}\left[|W_i^{(2)} \cap P_i| \leq \tfrac{1}{2}\sqrt[4]{\varepsilon}|P_i|\right] > 1 - n_i^{-1} \,. \tag{29}$$

For this we first partition P_i into sets of mutually predecessor disjoint pairs, i.e., $P_i = K_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} K_\ell$ such that for every $k \in [\ell]$ no vertex of X_i appears in two different pairs in K_k, and moreover no two pairs in K_k contain two vertices that have a common predecessor. Theorem 11 applied to the following graph asserts that there is such a partition with almost equally sized classes K_k: Let $\mathcal{P}$ be the graph on vertex set P_i with edges between exactly those pairs $\{y_1, y_2\}, \{y_1', y_2'\} \in P_i$ which have either $\{y_1, y_2\} \cap \{y_1', y_2'\} \neq \emptyset$ or $\big(N_H^-(y_1) \cup N_H^-(y_2)\big) \cap \big(N_H^-(y_1') \cup N_H^-(y_2')\big) \neq \emptyset$. This graph has maximum degree $\Delta(\mathcal{P}) < 2a\Delta(H) n_i \leq 2a(\xi n / \log n) n_i$. Hence Theorem 11 gives a partition $K_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} K_\ell$ of P_i into stable sets with $|K_k| \geq \lfloor |P_i| / \big(\Delta(\mathcal{P}) + 1\big) \rfloor \geq \log n / (8a\xi\kappa r)$ for all $k \in [\ell]$, where we used $|P_i| \geq n_i^2/4$.

Now fix $k \in [\ell]$ and consider the random variable $K_k' := K_k \cap W_i^{(2)}$. Our goal now is to show

$$\mathbb{P}\left[|K_k'| > \tfrac{1}{2}\sqrt[4]{\varepsilon}|K_k|\right] \leq n_i^{-3} \,, \tag{30}$$

as this together with another union bound over $k \in [\ell]$ with $\ell < n_i^2$ implies (29). We shall first bound the probability that some fixed pair $\{y, z\} \in K_k$ gets moved to $W_i^{(2)}(t)$ (and hence to K_k') at some time t.

For a pair $\{y,z\} \in K_k$ and $t \in [T]$ let $\mathcal{C}o_{t,y,z}$ denote the event that $|\deg_{\omega,t}(y,z) - \deg_{\omega,t-1}(y,z)| \le \varepsilon n_i$.

Claim 25 *Let $\{y,z\} \in K_k$ and t be such that $\{y,z\} \in W_i^{(2)}(t)$. Then there is $t' \le t$ such that $\mathcal{C}o_{t',y,z}$ does not hold.*

Proof of Claim 25. Assume for contradiction that $\{y,z\} \in W_i^{(2)}(t)$ and that $\mathcal{C}o_{t',y,z}$ holds for all $t' \le t$. For all time steps t' with $x_{t'} \notin N^-(y) \cup N^-(z)$ we have $\deg_{\omega,t'+1}(y,z) = \deg_{\omega,t'}(y,z)$. Combining this with the fact that $|N^-(y) \cup N^-(z)| \le 2a$ and using that $|d_{\omega,t}(X_i,V_i)^2 - \delta^{2a}| \le 2\sqrt{\varepsilon}\delta^{2a}$ by (26) we obtain

$$|\deg_{\omega,t}(y,z) - d_{\omega,t}(X_i,V_i)^2 n_i| \le |\deg_{\omega,t}(y,z) - \delta^{2a} n_i| + |d_{\omega,t}(X_i,V_i)^2 - \delta^{2a}|\, n_i$$
$$\le 2a\varepsilon n_i + 2\sqrt{\varepsilon}\delta^{2a} n_i \overset{(7)}{<} \sqrt[4]{\varepsilon} n_i\,.$$

In other words, $\{y,z\} \notin W_i^{(2)}(t)$, which is the desired contradiction. □

Moreover, if $\mathcal{C}o_{t',y,z}$ holds for all $t' < t$ then

$$|\deg_{\omega,t}(y,z) - \deg_{\omega,0}(y,z)| \le \big(\pi(t,y) + \pi(t,z)\big)\varepsilon n_i\,.$$

Recall that $\deg_{\omega,t}(y,z) = \delta^{a-\pi(t,y)}\delta^{a-\pi(t,z)}|C_{t,y} \cap C_{t,z}|$ and in particular (since $y,z \in X_i \setminus S_i$) $\deg_{\omega,0}(y,z) = \delta^{2a} n_i$. Hence

$$|C_{t,y} \cap C_{t,z}| \ge (\delta^{\pi(t,y)+\pi(t,z)} - \varepsilon\delta^{\pi(t,y)+\pi(t,z)-2a}(\pi(t,y)+\pi(t,z)))n_i \overset{(7)}{\ge} \varepsilon n_i\,. \tag{31}$$

We now claim that

$$\mathbb{P}[\mathcal{C}o_{t,y,z} \,|\, \mathcal{C}o_{t',y,z} \text{ for all } t' < t] \ge 1 - \frac{4\varepsilon}{\gamma}\,. \tag{32}$$

This is obvious if $x_t \notin N^-(y) \cup N^-(z)$. So assume we are about to embed an $x_t \in N^-(y) \cup N^-(z)$, which happens to be in X_j. Then $\varphi(x_t)$ is chosen randomly among at least $(\gamma/2)n_j$ vertices of V_j by Lemma 23. Out of those at most $2\varepsilon n_j$ vertices $v \in V_j$ have

$$\big|\deg(v, C_{t,y} \cap C_{t,z}) - d(V_i,V_j) \cdot |C_{t,y} \cap C_{t,z}|\big| > \varepsilon n_i$$

because $|C_{t,y} \cap C_{t,z}| \ge \varepsilon n_i$ by (31) and $G[V_i,V_j]$ is ε-regular. For every other choice of $\varphi(x_t) = v \in V_j$ we have

$$\begin{aligned}
&\big|\deg_{\omega,t+1}(y,z) - \deg_{\omega,t}(y,z)\big| \\
&\quad = \big|\omega_{t+1}(y)\omega_{t+1}(z)\deg(v, C_{t,y} \cap C_{t,z}) - \omega_t(y)\omega_t(z) \cdot |C_{t,y} \cap C_{t,z}|\big| \\
&\quad = \omega_{t+1}(y)\omega_{t+1}(z) \cdot \big|\deg(v, C_{t,y} \cap C_{t,z}) - \delta \cdot |C_{t,y} \cap C_{t,z}|\big| \\
&\quad = \omega_{t+1}(y)\omega_{t+1}(z) \cdot \big|\deg(v, C_{t,y} \cap C_{t,z}) - d(V_i,V_j) \cdot |C_{t,y} \cap C_{t,z}|\big| \\
&\quad \le \omega_{t+1}(y)\omega_{t+1}(z) \cdot \varepsilon n_i \le \varepsilon n_i\,.
\end{aligned}$$

Thus at most $2\varepsilon n_j$ out of $(\gamma/2)n_j$ choices for $\varphi(x_t)$ will result in $\overline{\mathcal{C}o_{t,y,z}}$, which implies (32), as claimed.

Finally, in order to show concentration, we will apply Lemma 19. For this purpose observe that by the construction of K_k for each time step $t \in [T]$ the embedding of x_t changes the co-degree of at most one pair in K_k, which we denote by $\{y_t, z_t\}$ if present. That is, $x_t \in N^-(y_t) \cup N^-(z_t)$. Now let $T_k \subseteq [T]$ be the set of time steps t with $\{y_t, z_t\}$ in K_k, i.e., let T_k be the set of time steps which actually change the co-degree of a pair in K_k. Since $|N^-(y) \cup N^-(z)| \leq 2a$ for every pair $\{y, z\} \in K_k$ we have $|T_k| \leq 2a|K_k|$. We define the following 0-1-variables $\mathcal{A}(t)$ for $t \in T_k$: Let $\mathcal{A}(t) = 1$ if and only if $\mathcal{C}o_{t',y_t,z_t}$ holds for all $t' \in [t-1] \cap T_k$ but not for $t' = t$. Recall that every $t \in T_k$ changes the co-degree of at most one pair in K_k. Claim 25 then implies that

$$\mathcal{A} := \sum_{t \in T_k} \mathcal{A}(t) = |\{(y, z) \in K_k : \mathcal{C}o_{t,y,z} \text{ does not hold for at least one } t \in T_k\}| \geq |K'_k| .$$

Moreover, for any $t' < t$ with $\{y_{t'}, z_{t'}\} = \{y_t, z_t\}$ and $\mathcal{A}(t') = 1$ we have $\mathcal{A}(t) = 0$ by definition. Hence, for any $t \in T_k$ and $J \subseteq [t] \cap T_k$ we have

$$\mathbb{P}\left[\mathcal{A}(t) = 1 \,\middle|\, \begin{array}{l} \mathcal{A}(t') = 1 \text{ for all } t' \in J \\ \mathcal{A}(t') = 0 \text{ for all } t' \in ([t] \cap T_k) \setminus J \end{array}\right] \leq 4\varepsilon/\gamma$$

by (32). Now either $|T_k| < 16a\varepsilon|K_k|/\gamma$ and thus $\mathcal{A} < 16a\varepsilon|K_k|/\gamma$ by definition. Or $|T_k| \geq 16a\varepsilon|K_k|/\gamma$ and

$$\mathbb{P}\left[\mathcal{A} \geq \frac{16a\varepsilon}{\gamma}|K_k|\right] \leq \mathbb{P}\left[\mathcal{A} \geq \frac{8\varepsilon}{\gamma}|T_k|\right] \leq \exp\left(-\frac{4\varepsilon}{3\gamma}|T_k|\right) \leq n_i^{-3},$$

by Lemma 19, where the last inequality follows from

$$\frac{4\varepsilon}{3\gamma}|T_k| \geq \frac{64a\varepsilon^2}{3\gamma^2}|K_k| \geq \frac{8\varepsilon^2 \log n}{3\gamma^2 \xi \kappa r} \overset{(9)}{\geq} 3 \log n_i \,.$$

Since $|K'_k| \leq \mathcal{A}$ and $16a\varepsilon/\gamma < \frac{1}{2}\sqrt[4]{\varepsilon}$ by (7) we obtain (30) as desired. $\square$

We have now established that the auxiliary graph $F_i(t)$ for the embedding of X_i into V_i is weighted regular for all times $t \leq T$ with positive probability. The following lemma states that no critical set ever gets large in this case, i.e., if all auxiliary graphs remain weighted regular, then the RGA terminates successfully.

Lemma 26
For every $t \leq T$ and $i \in [r]$ we have: $\mathcal{R}_i(t)$ implies that $|Q_i(t)| \leq \varepsilon' n_i$. In particular, $\mathcal{R}_i$ for all $i \in [r]$ implies that the RGA completes the Embedding Stage *successfully.*

Proof. The idea of the proof is the following. Vertices only become critical because their available candidate set is significantly smaller than the average available candidate set. In other words, the weighted density between the set of critical vertices and $V_i^{Free}(t)$ deviates significantly from the weighted density of the auxiliary graph. Since the auxiliary graph is weighted regular it follows that there cannot be many critical vertices.

Indeed, assume for contradiction that there is $i \in [r]$ and $t \leq T$ with $|Q_i(t)| > \varepsilon' n_i$ and such that $F_i(t)$ is weighted ε'-regular. Let $x \in Q_i(t)$ be an arbitrary critical vertex. Then x is an ordinary vertex and the available (ordinary) candidate set $A^{\mathrm{o}}_{t,x} = C_{t,x} \cap V_i^{\mathrm{o}} \cap V_i^{\mathit{Free}}(t)$ of x got small, that is,

$$|C_{t,x} \cap V_i^{\mathrm{o}} \cap V_i^{\mathit{Free}}(t)| \overset{(18)}{<} \gamma n_i \overset{(4)}{\leq} \frac{\mu}{20}\delta^a n_i \, .$$

In the language of the auxiliary graph this means that

$$\deg_{\omega,t}(x, V_i^{\mathrm{o}} \cap V_i^{\mathit{Free}}(t)) = \omega_t(x)|C_{t,x} \cap V_i^{\mathrm{o}} \cap V_i^{\mathit{Free}}(t)| \leq \frac{\mu}{20}\delta^a n_i.$$

Moreover $|V_i^{\mathrm{o}} \cap V_i^{\mathit{Free}}(t)| \geq |V_i^{\mathit{Free}}(t)| - |V_i^{\mathrm{s}}| \geq \frac{9}{10}\mu n_i \geq \varepsilon' n_i$. This implies

$$d_{\omega,t}(Q_i(t), V_i^{\mathrm{o}} \cap V_i^{\mathit{Free}}(t)) \leq \frac{\mu/20\,\delta^a n_i}{9/10\,\mu n_i} = \frac{1}{18}\delta^a \, . \tag{33}$$

Since (26) and (33) imply that

$$d_{\omega,t}(X_i, V_i) - d_{\omega,t}(Q_i(t), V_i^{\mathrm{o}} \cap V_i^{\mathit{Free}}(t)) \geq \frac{1}{2}\delta^a - \frac{1}{18}\delta^a > \varepsilon',$$

but $F_i(t)$ is weighted ε'-regular we conclude that $|Q_i(t)| < \varepsilon' n_i$. □

Theorem 6 is now immediate from the following lemma.

Lemma 27
If we apply the RGA in the setting of Theorem 6, then with probability at least $2/3$ the event $\mathcal{R}_i$ holds for all $i \in [r]$ and the RGA finds an embedding of H' into G (obeying the R-partitions of H and G and the image restrictions).

Proof of Lemma 27. Let C, a, Δ_R, κ and δ, c, μ be given. Set the constants $\gamma, \varepsilon, \alpha$ as in (4)-(8). Let r be given and choose n_0, ξ as in (9)-(10). Further let R be a graph of order r with $\Delta(R) < \Delta_R$ and let G, H, H' have the required properties. Run the RGA with these settings. The Initialisation succeeds with probability at least $5/6$ by Lemma 21. It follows from Lemma 24 that $\mathcal{R}_i$ occurs for all $i \in [r]$ with probability at least $5/6$. This implies that no critical set Q_i ever violates the bound (19) by Lemma 26. Thus the Embedding Stage also succeeds with probability $5/6$. We conclude that the RGA succeeds with probability at least $2/3$. Thus an embedding φ of $H' = H[X_1' \cup \ldots \cup X_r']$ into G which maps X_i' into V_i exists. Moreover this embedding guarantees $\varphi(x) \in I(x)$ for all $x \in S_i \cap X_i'$ by definition of the algorithm. □

At the end of this section we want to point out that the minimum degree bound for H in Theorem 6 can be increased even further if we swap the order of the quantifiers. More precisely, for a fixed graph R we may choose ε such that almost spanning subgraphs of linear maximum degree can be embedded into a corresponding (ε, d)-regular R-partition.

Theorem 28
Given a graph R of order r and positive parameters a, κ, δ, μ there are $\varepsilon, \xi > 0$ such that the following holds. Assume that we are given

(a) a graph G with a κ-balanced (ε, δ)-regular R-partition $V(G) = V_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_r$ with $|V_i| =: n_i$ and

(b) an a-arrangeable graph H with maximum degree $\Delta(H) \leq \xi n$ (where $n = \sum n_i$), together with a corresponding R-partition $V(H) = X_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} X_r$ with $|X_i| \leq (1-\mu)n_i$.

Then there is an embedding $\varphi\colon V(H) \to V(G)$ such that $\varphi(X_i) \subseteq V_i$.

Proof (sketch). Theorem 28 is deduced along the lines of the proof of Theorem 6.

Once more the randomised greedy algorithm from Section 3.2 is applied. It finds an embedding of H into G if all auxiliary graphs $F_i(t)$ remain weighted regular throughout the Embedding Stage. This in turn happens if each auxiliary graph $F_i(t)$ contains few pairs $\{x, y\} \in \binom{X_i}{2}$ whose weighted co-degree deviates from the expected value.

In the setting of Theorem 6 this is the case with positive probability as has been proven in Lemma 24: Inequality (29) states that the number of pairs with incorrect co-degree exceeds the bound of (28) with probability at most $1 - n_i^{-1}$. This particular argument is the only part of the proof of Theorem 6 that requires the degree bound of $\Delta(H) \leq \xi n/\log n$. We then used (29) and a union bound over $i \in [r]$ to show that all auxiliary graphs $F_i(t)$ remain weighted regular throughout the Embedding Stage with probability at least $5/6$. Since r can be large compared to all constants except n_0 we need the bound $1 - n_i^{-1}$ in (29).

In the setting of Theorem 28 however it suffices to replace this bound by a constant. More precisely, since we are allowed to choose ε depending on the order of R the proof of Lemma 24 becomes even simpler: Set ε, ξ small enough to ensure $8a\varepsilon/\gamma + 2a\kappa r\xi \leq \sqrt[4]{\varepsilon}/(6r)$. Note that inequality (32) then implies that the expected number of pairs $\{y, z\} \in \binom{X_i}{2}$ with incorrect co-degree is bounded by $2a\frac{4\varepsilon}{\gamma}\binom{n_i}{2} + a\Delta(H)n_i \leq \frac{\sqrt[4]{\varepsilon}}{6r}\binom{n_i}{2}$.

It follows from Markov's inequality and the union bound over all $i \in [r]$ that all auxiliary graphs $F_i(t)$ remain weighted regular throughout the Embedding Stage with probability at least $5/6$. Choosing ε sufficiently small we can thus guarantee that the randomised greedy algorithm successfully embeds H into G with positive probability. □

Using the classical approach of Chvatal, Rödl, Szemerédi, and Trotter [7] Theorem 28 easily implies that all a-arrangeable graphs have linear Ramsey numbers. This result has first been proven (using the approach of [7]) by Chen and Schelp [6].

4 The spanning case

In this section we prove our main result, Theorem 4. We use the randomised greedy algorithm and its analysis from Section 3 to infer that the almost spanning embedding found in Theorem 6 can in fact be extended to a spanning embedding. We shortly describe our strategy in Section 4.1 and establish a minimum degree bound for the auxiliary graphs in Section 4.2 before we give the proof of Theorem 4 in Section 4.3. We conclude this section with a sketch of the proof of Theorem 5 in Section 4.4.

4.1 Outline of the proof

Let G, H satisfy the conditions of Theorem 4. We first use Lemma 13 to order the vertices of H such that the arrangeability of the resulting order is bounded and its last μn vertices form a stable set W. We then run the RGA to embed the almost spanning subgraph $H' = H[X \setminus W]$ into G. The RGA is successful and the resulting auxiliary graphs $F_i(T)$ are all weighted regular (that is, $\mathcal{R}_i$ holds) with probability $2/3$ by Lemma 27.

It remains to extend the embedding of H' to an embedding of H. Since W is stable it suffices to find for each $i \in [r]$ a bijection between

$$L_i := X_i \setminus W$$

and $V_i^{\mathit{Free}}(T)$ which respects the candidate sets, i.e., which maps x into $C_{T,x}$. Such a bijection is given by a perfect matching in $F_i^* := F_i(T)[L_i \cup V_i^{\mathit{Free}}(T)]$, which is the subgraph of $F_i(T)$ induced by the vertices left after the Embedding Phase of the RGA.

By Lemma 18 balanced weighted regular pairs with an appropriate minimum degree bound have perfect matchings. Now, $(L_i, V_i^{\mathit{Free}})$ is a subpair of a weighted regular pair and thus weighted regular itself by Proposition 15. Hence our main goal is to establish a minimum degree bound for $(L_i, V_i^{\mathit{Free}})$. More precisely we shall explain in Section 4.2 that it easily follows from the definition of the RGA that vertices in L_i have the appropriate minimum degree if $\mathcal{R}_i$ holds.

Proposition 29
Run the RGA in the setting of Theorem 4 and assume that $\mathcal{R}_j$ holds for all $j \in [r]$. Then every $x \in L_i$ has

$$\deg_{F_i(T)}(x, V_i^{\mathit{Free}}(T)) \geq 3\sqrt{\varepsilon'} n_i.$$

For vertices in V_i^{Free} on the other hand this is not necessarily true. But it holds with sufficiently high probability. This is also proved in Section 4.2.

Lemma 30
Run the RGA in the setting of Theorem 4 and assume that $\mathcal{R}_j$ holds for all $j \in [r]$. Then we have

$$\mathbb{P}\left[\forall i \in [r],\ \forall v \in V_i^{\mathit{Free}}(T) : \deg_{F_i(T)}(v, L_i) \geq 3\sqrt{\varepsilon'} n_i\right] \geq \frac{2}{3}\,.$$

4.2 Minimum degree bounds for the auxiliary graphs

In this section we prove Proposition 29 and Lemma 30. For the former we need to show that vertices $x \in L_i$ have an appropriate minimum degree in F_i^*, which is easy.

Proof of Proposition 29. Since $\mathcal{R}_j$ holds for all $j \in [r]$ the RGA completed the EMBEDDING STAGE successfully by Lemma 26. Note that all $x \in L_i$ did not get embedded yet. Thus

$$\deg_{F_i(T)}(x, V_i^{Free}(T)) = |A_{T,x}| \geq |A^{s}_{T,x}| \geq \tfrac{7}{10}\gamma n_i \overset{(6)}{\geq} 3\sqrt{\varepsilon'} n_i\,,$$

for every $x \in L_i$ by Lemma 23. □

Lemma 30 claims that vertices in $V_i^{Free}(T)$ with positive probability also have a sufficiently large degree in F_i^*. We sketch the idea of the proof.

Let $x \in L_i$ and $v \in V_i^{Free}(T)$ for some $i \in [r]$. Recall that there is an edge $xv \in E(F_i(T))$ if and only if $\varphi(N^-(x)) \subseteq N_G(v)$. So we aim at lower-bounding the probability that $\varphi(N^-(x)) \subseteq N_G(v)$ for many vertices $x \in L_i$.

Now let $y \in N^-(x)$ be a predecessor of x. Recall that y is randomly embedded into $A(y)$, as defined in (17). Hence the probability that y is embedded into $N_G(v)$ is $|A(y) \cap N_G(v)|/|A(y)|$. Our goal will now be to show that these fractions are bounded from below by a constant for all predecessors of many vertices $x \in L_i$, which will then imply Lemma 30. To motivate this constant lower bound observe that a random subset A of X_j satisfies $|A \cap N_G(v)|/|A| = |N_G(v) \cap V_j|/|V_j|$ in expectation, and the right hand fraction is bounded from below by $\delta/2$ by (3). For this reason we call the vertex v *likely for* $y \in X_j$ and say that $\mathcal{A}_v(y)$ holds, if

$$\frac{|A(y) \cap N_G(v)|}{|A(y)|} \geq \frac{2}{3}\frac{|V_j \cap N_G(v)|}{|V_j|}\,.$$

Hence it will suffice to prove that for every $v \in V_i$ there are many $x \in L_i$ such that v is likely for all $y \in N^-(x)$.

We will focus on the last λn_i vertices x in $L_i \setminus N(S)$ (i.e., on vertices $x \in L_i^*$) as we have a good control over the embedding of their predecessors (who are in $X^* \setminus S$). Note that there indeed are λn_i vertices in $L_i \setminus N(S)$ as $\mu n_i - \alpha n_i \geq \lambda n_i$. For $i \in [r]$ and $v \in V_i$ we define

$$L_i^*(v) := \{x \in L_i^* : \mathcal{A}_v(y) \text{ holds for all } y \in N^-(x)\}\,.$$

Our goal is to show that a positive proportion of the vertices in L_i^* will be in $L_i^*(v)$. The following lemma makes this more precise.

Lemma 31
We run the RGA in the setting of Theorem 4 and assume that $\mathcal{R}_j$ holds for all $j \in [r]$. Then

$$\mathbb{P}\left[\forall i \in [r], \forall v \in V_i : |L_i^*(v)| \geq 2^{-a^2-1}|L_i^*|\right] \geq \frac{5}{6}\,.$$

Lemma 31 together with the subsequent lemma will imply Lemma 30.

Lemma 32
Run the RGA in the setting of Theorem 4 and assume that $\mathcal{R}_j$ holds for all $j \in [r]$ and that $|L_i^*(v)| \geq 2^{-a^2-1}|L_i^*|$. Then we have

$$\mathbb{P}\left[\forall i \in [r],\ \forall v \in V_i^{Free}(T) : \deg_{F_i(T)}(v, L_i) \geq 3\sqrt{\varepsilon'} n_i\right] \geq \frac{5}{6}\,.$$

Proof of Lemma 32. Let $i \in [r]$ and $v \in V_i$ be arbitrary and assume that the event of Lemma 31 occurs, this is, assume that we do have $|L_i^*(v)| \geq 2^{-a^2-1}|L_i^*|$. We claim that v almost surely has high degree in $F_i(T)$ in this case.

Claim 33 *If $|L_i^*(v)| \geq 2^{-a^2-1}|L_i^*|$, then*

$$\mathbb{P}\left[\deg_{F_i(T)}(v, L_i) \geq 3\sqrt{\varepsilon'} n_i\right] \geq 1 - \frac{1}{n_i^2}\,.$$

This claim, together with a union bound over all $i \in [r]$ and $v \in V_i$, implies that

$$\mathbb{P}\left[\forall i \in [r], \forall v \in V_i : \deg_{F_i(T)}(v, L_i) \geq 3\sqrt{\varepsilon'} n_i\right] \geq \frac{5}{6}$$

if $|L_i^*(v)| \geq 2^{-a^2-1}|L_i^*|$ for all $i \in [r]$ and all $v \in V_i$. It remains to establish the claim.

Proof of Claim 33. Let $x \in L_i^*(v)$. Recall that $xv \in E(F_i)$ if and only if $\varphi(y) \in N_G(v)$ for all $y \in N^-(x)$. If the events $[\varphi(y) \in N_G(v)]$ were independent for all $y \in N^-(L_i^*(v))$ we could apply a Chernoff bound to infer that almost surely a linear number of the vertices $x \in L_i^*(v)$ is such that $[\varphi(y) \in N_G(v)]$ for all $y \in N^-(x)$. However, the events might be far from independent: just imagine two vertices x, x' sharing a predecessor y. We address this issue by partitioning the vertices into classes that do not share predecessors. We then apply Lemma 20 to those classes to finish the proof of the claim. Here come the details.

We partition $L_i^*(v)$ into predecessor disjoint sets. To do so we construct an auxiliary graph on vertex set $L_i^*(v)$ that has an edge xx' for exactly those vertices $x \neq x'$ that share at least one predecessor in H. As H is a-arrangeable, the maximum degree of this auxiliary graph is bounded by $a\Delta(H) - 1$. Hence we can apply Theorem 11 to partition the vertices of this auxiliary graph into stable sets $K_1 \dot\cup \ldots \dot\cup K_b$ with

$$|K_\ell| \geq \frac{|L_i^*(v)|}{a\Delta(H)} \geq \frac{2^{-a^2-1}\lambda\sqrt{n}}{a\,\kappa\,r}\log n \overset{(10)}{\geq} 48\left(\frac{3}{\delta}\right)^a \log n_i \tag{34}$$

for $\ell \in [b]$, where the second inequality follows from our assumption $|L_i^*(v)| \geq 2^{-a^2-1}|L_i^*|$ and from $|L_i^*| = \lambda n_i \geq \lambda n/(\kappa r)$ and $\Delta(H) \leq \sqrt{n/\log n}$.

Those sets are predecessor disjoint in H. We now want to apply Lemma 20. Let $\mathcal{I} = \{N^-(x) : x \in K_\ell\}$. The sets in $\mathcal{I}$ are pairwise disjoint and have at most a elements each. Name the elements of $\bigcup_{I\in\mathcal{I}} I = \{y_1, \ldots, y_s\}$ (with $s = |\bigcup_{I\in\mathcal{I}} I|$) in ascending order with respect to the arrangeable ordering. Furthermore, let $\mathcal{A}_k$ be a random variable which is 1 if and only if y_k gets embedded into $N_G(v)$. By the definition of $L_i^*(v)$, the event $\mathcal{A}_v(y_k)$ holds for each $k \in [s]$. It follows from the definition of $\mathcal{A}_v(y_k)$ that

$$\mathbb{P}[\mathcal{A}_k = 1] = \mathbb{P}[\varphi(y_k) \in N_G(v)] = \frac{|A(y_k) \cap N_G(v)|}{|A(y_k)|} \geq \frac{2}{3}\frac{|N_G(v) \cap V_j|}{|V_j|} \overset{(3)}{\geq} \frac{\delta}{3}\,.$$

This lower bound on the probability of $\mathcal{A}_k = 1$ remains true even if we condition on other events $\mathcal{A}_j = 1$ (or their complements $\mathcal{A}_j = 0$), because in this calculation the lower bound relies solely on $|A(y_k) \cap N_G(v)|/|A(y_k)|$, which is at least $\delta/3$ for all $k \in [m]$ regardless of the embedding of other y_j. Hence,

$$\mathbb{P}\left[\mathcal{A}_k = 1 \middle| \begin{matrix} \mathcal{A}_j = 1 \text{ for all } j \in J \\ \mathcal{A}_j = 0 \text{ for all } j \in [k-1] \setminus J \end{matrix}\right] \geq \frac{\delta}{3}$$

for every k and every $J \subseteq [k-1]$ (this is stronger than the condition required by Lemma 20). By Lemma 20, we have

$$\begin{aligned}
\mathbb{P}&\left[\left|\left\{x \in K_\ell : \varphi(N^-(x)) \subseteq N_G(v)\right\}\right| \geq \frac{1}{2}\left(\frac{\delta}{3}\right)^a |K_\ell|\right] \\
&= \mathbb{P}\left[\left|\left\{I \in \mathcal{I} : \mathcal{A}_i = 1 \text{ for all } i \in I\right\}\right| \geq \frac{1}{2}\left(\frac{\delta}{3}\right)^a |K_\ell|\right] \\
&\geq 1 - 2\exp\left(-\frac{1}{12}\left(\frac{\delta}{3}\right)^a |K_\ell|\right) \\
&\overset{(34)}{\geq} 1 - 2\exp(-4\log n_i) = 1 - 2 \cdot n_i^{-4}\,.
\end{aligned}$$

Applying a union bound over all $\ell \in [b]$ we conclude that

$$\begin{aligned}
\deg_{F_i(T)}(v, L_i) &\geq \left|\left\{x \in L_i^*(v) : \varphi(N^-(x)) \subseteq N_G(v)\right\}\right| \\
&\geq \sum_{\ell \in [b]} \frac{1}{2}\left(\frac{\delta}{3}\right)^a |K_\ell| = \frac{1}{2}\left(\frac{\delta}{3}\right)^a |L_i^*(v)| \geq \frac{\delta^a}{2 \cdot 2^{a^2+1} 3^a}\lambda n_i \overset{(6)}{\geq} 3\sqrt{\varepsilon'} n_i
\end{aligned}$$

with probability at least $1 - 2n_i \exp(-4\log n_i) \geq 1 - n_i^{-2}$. □

This concludes the proof of the lemma. □

The remainder of this section is dedicated to the proof of Lemma 31. This proof will use similar ideas as the proof of Lemma 32. This time, however, we are not only interested in the predecessors of $x \in L_i^*$ but in the predecessors of the predecessors. We call those *predecessors of second order* and say two vertices x, x' are *predecessor disjoint of second order* if $N^-(x) \cap N^-(x') = \emptyset$ and $N^-(N^-(x)) \cap N^-(N^-(x')) = \emptyset$.

To prove Lemma 31, we have to show that for any vertex $v \in V_i$ many vertices x in L_i^* are such that all their predecessors $y \in N^-(x)$ are likely for v. Note that $x \in L_i^*$ implies that $y \in N^-(x)$ gets embedded into the special candidate set $C^{\mathrm{s}}_{t(y),y}$.

It depends only on the embedding of the vertices in $N^-(y)$ whether a given vertex $v \in V_i$ is likely for y or not. Therefore, we formulate an event $\mathcal{B}_{v,x}(z)$, which, if satisfied for all $z \in N^-(y)$, will imply $\mathcal{A}_v(y)$ as we will show in the next proposition. Recall that $C^{\mathrm{s}}_{1,y} = V^{\mathrm{s}}_j$ for $y \in N^-(L^*_i) \subseteq X^*$ and $C^{\mathrm{s}}_{t(z)+1,y} = C^{\mathrm{s}}_{t(z),y} \cap N_G(\varphi(z))$. For $x \in L^*_i$ and $z \in N^-(N^-(x))$ let $\mathcal{B}_{v,x}(z)$ be the event that

$$\left| \frac{|C^{\mathrm{s}}_{t(z),y} \cap N_G(v)|}{|C^{\mathrm{s}}_{t(z),y}|} - \frac{|C^{\mathrm{s}}_{t(z)+1,y} \cap N_G(v)|}{|C^{\mathrm{s}}_{t(z)+1,y}|} \right| \leq \frac{2\varepsilon}{\delta - \varepsilon}$$

for all $y \in N^-(x)$.

Proposition 34
Let $i \in [r]$, $v \in V_i$, $x \in L^*_i$, and $z \in N^-(N^-(x))$, then

$$\mathbb{P}\left[\mathcal{B}_{v,x}(z) \middle| \mathcal{B}_{v,x}(z') \text{ for all } z' \in N^-(N^-(x)) \text{ with } t(z') < t(z)\right] \geq 1/2.$$

This remains true if we additionally condition on other events $\mathcal{B}_{v,\widetilde{x}}(\widetilde{z})$ (or their complements) with $\widetilde{z} \in N^-(N^-(\widetilde{x}))$ for $\widetilde{x} \in L^*_i$, as long as x and $\widetilde{x}$ are predecessor disjoint of second order.
Moreover, if $\mathcal{B}_{v,x}(z)$ occurs for all $z \in N^-(N^-(x))$, then $\mathcal{A}_v(y)$ occurs for all $y \in N^-(x)$.

Proof of Proposition 34. Let $x \in L^*_i$ and let $z \in N^-(N^-(x))$ lie in X_ℓ. Further assume that $\mathcal{B}_{v,x}(z')$ holds for all $z' \in N^-(N^-(x))$ with $t(z') < t(z)$. For $y \in N^-(x)$ let $j(y)$ be such that $y \in X_{j(y)}$. Then $\mathcal{B}_{v,x}(z')$ for all $z' \in N^-(y)$ with $t(z') < t(z)$ implies

$$\begin{aligned} \frac{|C^{\mathrm{s}}_{t(z),y} \cap N_G(v)|}{|C^{\mathrm{s}}_{t(z),y}|} &\geq \frac{|C^{\mathrm{s}}_{1,y} \cap N_G(v)|}{|C^{\mathrm{s}}_{1,y}|} - \frac{2\varepsilon \cdot a}{\delta - \varepsilon} \\ &= \frac{|V_{j(y)} \cap N_G(v)|}{|V_{j(y)}|} - \frac{2\varepsilon \cdot a}{\delta - \varepsilon} \overset{(3)}{\geq} \frac{\delta}{2} - \frac{2\varepsilon \cdot a}{\delta - \varepsilon} \end{aligned}$$

where the identity $C_{1,y} = V_{j(y)}$ is due to $y \notin S$. Hence $|C^{\mathrm{s}}_{t(z),y} \cap N_G(v)| \geq \varepsilon n_{j(\ell)}$ by (16) and our choice of constants. Now fix a $y \in N^-(x)$. As $(V_{j(y)}, V_\ell)$ is an ε-regular pair all but at most $4\varepsilon n_\ell$ vertices $w \in A_{t(z),z} \subseteq V_\ell$ simultaneously satisfy

$$\begin{aligned} \left| \frac{\left|N_G\big(w, C^{\mathrm{s}}_{t(z),y} \cap N_G(v)\big)\right|}{|C^{\mathrm{s}}_{t(z),y} \cap N_G(v)|} - d(V_{j(y)}, V_\ell) \right| &\leq \varepsilon\,, \text{ and} \\ \left| \frac{|N_G(w, C^{\mathrm{s}}_{t(z),y})|}{|C^{\mathrm{s}}_{t(z),y}|} - d(V_{j(y)}, V_\ell) \right| &\leq \varepsilon\,. \end{aligned}$$

Hence, all but at most $4\varepsilon a n_\ell$ vertices in V_ℓ satisfy the above inequalities for all $y \in N^-(x)$. If $\varphi(z) = w$ for a vertex w that satisfies the above inequalities for all $y \in N^-(x)$ we have

$$\left| \frac{|C^{\mathrm{s}}_{t(z)+1,y} \cap N_G(v)|}{|C^{\mathrm{s}}_{t(z),y} \cap N_G(v)|} - \frac{|C^{\mathrm{s}}_{t(z)+1,y}|}{|C^{\mathrm{s}}_{t(z),y}|} \right| \leq 2\varepsilon\,.$$

This implies $B_{v,x}(z)$ as

$$\left|\frac{|C^{\mathrm{s}}_{t(z)+1,y}\cap N_G(v)|}{|C^{\mathrm{s}}_{t(z)+1,y}|}-\frac{|C^{\mathrm{s}}_{t(z),y}\cap N_G(v)|}{|C^{\mathrm{s}}_{t(z),y}|}\right|\leq 2\varepsilon\frac{|C^{\mathrm{s}}_{t(z),y}\cap N_G(v)|}{|C^{\mathrm{s}}_{t(z)+1,y}|}$$
$$\leq 2\varepsilon\frac{|C^{\mathrm{s}}_{t(z),y}|}{|C^{\mathrm{s}}_{t(z)+1,y}|}\overset{(15)}{\leq}\frac{2\varepsilon}{\delta-\varepsilon}\,.$$

Since $\varphi(z)$ is chosen randomly from $A(z)\subseteq A_{t(z),z}$ with $|A(z)|\geq(\gamma/2)n_\ell$ by Lemma 23, we obtain

$$\mathbb{P}\left[\mathcal{B}_{v,x}(z)\middle|\mathcal{B}_{v,x}(z')\text{ for all }z'\in N^-(N^-(x)),\,t(z')<t(z)\right]\geq 1-\frac{4\varepsilon a n_\ell}{(\gamma/2)n_\ell}\geq\frac{1}{2}\,.$$

Note that this probability follows alone from the ε-regularity of the pairs $(V_{j(y)},V_\ell)$ and the fact that $A(z)$ and $C^{\mathrm{s}}_{t(z),y}\cap N_G(v)$ are large. If x and $\widetilde{x}$ are predecessor disjoint of second order the outcome of the event $\mathcal{B}_{v,\widetilde{x}}(\widetilde{z})$ for $\widetilde{z}\in N^-(N^-(\widetilde{x}))$ does not influence those parameters. We can therefore condition on other events $\mathcal{B}_{v,\widetilde{x}}(\widetilde{z})$ as long as x and $\widetilde{x}$ are predecessor disjoint of second order.

It remains to show the second part of the proposition, that v is likely for all $y\in N^-(x)$ if $\mathcal{B}_{v,x}(z)$ holds for all $z\in N^-(N^-(x))$. Again let $x\in L^*_i$ and let $y\in N^-(x)$ lie in X_j. Recall that condition (13) in the definition of the RGA guarantees

$$\left|\frac{|V^{\mathrm{s}}_j\cap N_G(v)|}{|V^{\mathrm{s}}_j|}-\frac{|V_j\cap N_G(v)|}{|V_j|}\right|\leq\varepsilon\,.$$

Moreover, $\mathcal{B}_{v,x}(z)$ for all $z\in N^-(y)$, $C^{\mathrm{s}}_{1,y}=V^{\mathrm{s}}_j$ (as $y\notin S$) and the fact that $|N^-(y)|\leq a$ imply

$$\left|\frac{|C^{\mathrm{s}}_{t(y),y}\cap N_G(v)|}{|C^{\mathrm{s}}_{t(y),y}|}-\frac{|V^{\mathrm{s}}_j\cap N_G(v)|}{|V^{\mathrm{s}}_j|}\right|\leq\frac{2\varepsilon\cdot a}{\delta-\varepsilon}\,.$$

As $2\varepsilon a/(\delta-\varepsilon)+\varepsilon\leq\delta/36\leq(\delta/18)|V_j\cap N_G(v)|/|V_j|$ we conclude that

$$\frac{|C^{\mathrm{s}}_{t(y),y}\cap N_G(v)|}{|C^{\mathrm{s}}_{t(y),y}|}\geq\frac{17}{18}\frac{|V_j\cap N_G(v)|}{|V_j|}\tag{35}$$

for all $y\in N^-(x)$ if $\mathcal{B}_{v,x}(z)$ for all $z\in N^-(N^-(x))$. Equation (35) in turn implies $\mathcal{A}_v(y)$ as only few vertices get embedded into V^{s} thus making $C^{\mathrm{s}}_{t,y}\approx A^{\mathrm{s}}_{t,y}$. More precisely, by

Lemma 23 we have

$$\begin{aligned}\frac{|A(y)\cap N_G(v)|}{|A(y)|} &\geq \frac{|A^{\mathrm{s}}_{t(y),y}\cap N_G(v)| - |A^{\mathrm{s}}_{t(y),y}\setminus A(y)|}{|A^{\mathrm{s}}_{t(y),y}|}\\ &\geq \frac{|C^{\mathrm{s}}_{t(y),y}\cap N_G(v)| - |X^*_j| - |Q_j(t(y))| - |A^{\mathrm{s}}_{t(y),y}\setminus A(y)|}{|C^{\mathrm{s}}_{t(y),y}|}\\ &\geq \frac{|C^{\mathrm{s}}_{t(y),y}\cap N_G(v)|}{|C^{\mathrm{s}}_{t(y),y}|} - \frac{\delta}{18}\\ &\overset{(35)}{\geq} \left(\frac{17}{18}-\frac{2}{18}\right)\frac{|V_j\cap N_G(v)|}{|V_j|}\,. \qquad \square\end{aligned}$$

We have seen that $x \in L^*_i$ also lies in $L^*_i(v)$ if $\mathcal{B}_{v,x}(z)$ holds for all $z \in N^-(N^-(x))$. To prove Lemma 31 it therefore suffices to show that an arbitrary vertex v has a linear number of vertices $x \in L^*_i$ with $\mathcal{B}_{v,x}(z)$ for all $z \in N^-(N^-(x))$.

Proof of Lemma 31. Let $i \in [r]$ and $v \in V_i$ be arbitrary. We partition L^*_i into classes of vertices that are predecessor disjoint of second order. Observe that for every $x \in L^*_i$ we have

$$\begin{aligned}\left|\left\{x' \in L^*_i : \begin{matrix} N^-(x)\cap N^-(x') \neq \emptyset \text{ or} \\ N^-(N^-(x))\cap N^-(N^-(x')) \neq \emptyset\end{matrix}\right\}\right| &\leq (a\Delta(H))^2 \leq \frac{a^2 n}{\log^2 n}\\ &\overset{(10)}{<} \frac{\lambda n_i}{36\cdot 2^{a^2}\log n_i}\end{aligned}$$

as H is a-arrangeable. Recall that $|L^*_i| = \lambda n_i$. Therefore, Theorem 11 gives a partition $L^*_i = K_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} K_b$ with

$$|K_\ell| \geq 36\cdot 2^{a^2}\log n_i \tag{36}$$

for all $\ell \in [b]$ such that the vertices in K_ℓ are predecessor disjoint of second order.

Next we want to apply Lemma 20. Let $\ell \in [b]$ be fixed. We define $\mathcal{I} = \{N^-(N^-(x)) : x \in K_\ell\}$. These sets are pairwise disjoint and have at most a^2 elements each. Name the elements of $\bigcup_{I\in\mathcal{I}} I = \{z_1, \ldots, z_{|\cup_{I\in\mathcal{I}} I|}\}$ in ascending order with respect to the arrangeable ordering. Then for every $I \in \mathcal{I}$ and every $z_k \in I$ we have

$$\mathbb{P}\left[\mathcal{B}_{v,x}(z_k) \middle| \begin{matrix}\mathcal{B}_{v,x}(z_j) \text{ for all } z_j \in J, \\ \overline{\mathcal{B}_{v,x}}(z_j) \text{ for all } z_j \in \{z_1,\ldots,z_{k-1}\}\setminus J\end{matrix}\right] \geq \frac{1}{2}$$

for every $J \subseteq \{z_1, \ldots, z_{k-1}\}$ with $\{z_1, \ldots, z_{k-1}\} \cap I \subseteq J$ by Proposition 34. We set $K_\ell(v) := \{x \in K_\ell : \mathcal{B}_{v,x}(z) \text{ for all } z \in N^-(N^-(x))\}$ and apply Lemma 20 to derive

$$\begin{aligned}\mathbb{P}\left[|K_\ell(v)| \geq 2^{-a^2-1}|K_\ell|\right] &\geq 1 - 2\exp\left(-\frac{1}{12}2^{-a^2}|K_\ell|\right)\\ &\overset{(36)}{\geq} 1 - 2\exp\left(-3\log n_i\right) = 1 - 2\cdot n_i^{-3}\,.\end{aligned}$$

Note that we have $\bigcup_{\ell\in[b]} K_\ell(v) \subseteq L_i^*(v)$ as the following is true for every $x \in K_\ell$ by Proposition 34: $\mathcal{B}_{v,x}(z)$ for all $z \in N^-(N^-(x))$ implies $\mathcal{A}_v(y)$ for all $y \in N^-(x)$. Taking a union bound over all $\ell \in [b]$ we thus obtain that

$$\mathbb{P}\left[|L_i^*(v)| \geq 2^{-a^2-1}|L_i^*|\right] \geq 1 - b\cdot 2n_i^{-3} \geq 1 - \frac{2}{n_i^2}\,.$$

One further union bound over all $i \in [r]$ and $v \in V_i$ finishes the proof. □

4.3 Proof of Theorem 4

Putting everything together, we conclude that the RGA gives a spanning embedding of H into G with probability at least $1/3$. We now use Lemma 18, Proposition 29, and Lemma 30 to prove our main result.

Proof of Theorem 4. Let integers C, a, Δ_R, κ and $\delta, c > 0$ be given. Set $a' = 5a^2\kappa\Delta_R$ and $\mu = 1/(10a'(\kappa\Delta_R)^2)$. We invoke Theorem 6 with parameters C, a', Δ_R, κ and $\delta, c, \mu > 0$ to obtain $\varepsilon, \alpha > 0$. Let r be given and choose n_0 as in Theorem 6.

Now let R be a graph on r vertices with $\Delta(R) < \Delta_R$. And let G and H satisfy the conditions of Theorem 4, i.e., let G have the (ε,δ)-super-regular R-partition $V(G) = V_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_r$ and let H have a κ-balanced R-partition $V(H) = X_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} X_r$. Further let $\{x_1, \ldots, x_n\}$ be an a-arrangeable ordering of H. We apply Lemma 13 to find an a'-arrangeable ordering $\{x_1', \ldots, x_n'\}$ of H with a stable ending of order μn. Let $H' = H[\{x_1', \ldots, x_{(1-\mu)n}'\}]$ be the subgraph induced by the first $(1-\mu)n$ vertices of the new ordering. We take this ordering and run the RGA as described in Section 3.2 to embed H' into G. By Lemma 27 we have

$$\mathbb{P}\left[\text{RGA successful and } \mathcal{R}_i \text{ for all } i \in [r]\right] \geq \frac{2}{3}\,, \tag{37}$$

where $\mathcal{R}_i$ is the event that the auxiliary graph $F_i(t)$ is weighted ε'-regular for all $t \leq T$. Note that every image restricted vertex $x \in S_i \cap V(H')$ has been embedded into $I(x)$ by the definition of the RGA.

Now assume that $\mathcal{R}_i$ holds for all $i \in [r]$. It remains to embed the stable set $L_i = V(H) \setminus V(H')$. To this end we shall find in each $F_i^* := F_i(T)[L_i \mathbin{\dot\cup} V_i^{\mathit{Free}}(T)]$ a perfect matching, which defines a bijection from L_i to $V_i^{\mathit{Free}}(T)$ that maps every $x \in L_i$ to a vertex $v \in V_i^{\mathit{Free}}(T) \cap C_{T,x}$. Note that again $x \in S_i$ is embedded into $I(x)$ by construction.

Since $F_i(T)$ is weighted ε'-regular the subgraph F_i^* is weighted (ε'/μ)-regular by Proposition 15. Moreover

$$\mathbb{P}\left[\delta(F_i^*) \geq 3\sqrt{\varepsilon'} n_i \text{ for all } i \in [r]\right] \geq \frac{2}{3} \tag{38}$$

by Proposition 29 and Lemma 30. In other words, with probability at least $2/3$ all graphs $F_i^* = F_i(T)[L_i \mathbin{\dot\cup} V_i^{\mathit{Free}}]$ are balanced, bipartite graphs on $2\mu n_i$ vertices with $\deg(x) \geq 3\sqrt{\varepsilon'/\mu}(\mu n_i)$ for all $x \in L_i \cup V_i^{\mathit{Free}}$. Also note that $\omega(x) \geq \delta^a \geq \sqrt{\varepsilon'/\mu}$ for all $x \in L_i$ by definition of ε'. We conclude from Lemma 18 that F_i^* has a perfect matching if F_i^* has minimum degree at least $3\sqrt{\varepsilon'}n_i$. Hence, combining (37) and (38) we obtain that the RGA terminates successfully and all F_i^* have perfect matchings with probability at least $1/3$. Thus there is an almost spanning embedding of H' into G that can be extended to a spanning embedding of H into G. □

4.4 Proof of Theorem 5

We close this section by sketching the proof of Theorem 5, which is very similar to the proof of Theorem 4. We start by quickly summarising the latter. For two graphs G and H let the partitions $V = V_1 \cupdot \ldots \cupdot V_k$ and $X = X_1 \cupdot \ldots \cupdot X_k$ satisfy the requirements of Theorem 4. In order to find an embedding of H into G that maps the vertices of X_i onto V_i we proceeded in two steps. First we used a randomised greedy algorithm to embed an almost spanning part of H into G. This left us with sets $L_i \subseteq X_i$ and $V_i^{\mathit{Free}} \subseteq V_i$. We then found a bijection between the L_i and V_i^{Free} that completed the embedding of H.

More precisely, we did the following. We ran the randomised greedy algorithm from Section 3.2 and defined auxiliary graphs $F_i(t)$ on vertex sets $V_i \cupdot X_i$ that kept track of all possible embeddings at time t of the embedding algorithm. We showed that the randomised greedy embedding succeeds for the almost spanning subgraph if all the auxiliary graphs remain weighted regular (Lemma 26). This in turn happens with probability at least $2/3$ by Lemma 27. This finished stage one of the embedding (and also proved Theorem 6).

For the second stage of the embedding we assumed that stage one found an almost spanning embedding by time T and that all auxiliary graphs are weighted regular. We defined $F_i^*(T)$ to be the subgraph of $F_i(T)$ induced by $L_i \cupdot V_i^{\mathit{Free}}$. This subgraph inherits (some) weighted regularity from $F_i(T)$. Moreover, we showed that all $F_i^*(T)$ have a minimum degree which is linear in n_i with probability at least $2/3$ (see Proposition 29 and Lemma 30). Each $F_i^*(T)$ has a perfect matching in this case by Lemma 18. Those perfect matchings then gave the bijection of L_i onto V_i^{Free} that completed the embedding of H into G. We concluded that with probability at least $2/3$ the almost spanning embedding found by the randomised greedy algorithm in stage one can be extended to a spanning embedding of H into G.

For the proof of Theorem 5 we proceed in exactly the same way. Note that Theorem 4 and Theorem 5 differ only in the following aspects. The first allows a maximum degree of $\sqrt{n}/\log n$ for H while the latter extends this to $\Delta(H) \leq \xi n/\log n$. This does not come free of charge. Theorem 5 not only requires the R-partition of G to be super-regular but also imposes what we call the *tuple condition*, that every tuple of $a+1$ vertices in $V \setminus V_i$ have a linearly sized joint neighbourhood in V_i. We now sketch how one has to change the proof of Theorem 4 to obtain Theorem 5.

Again we proceed in two stages. The first of those, which gives the almost spanning embedding, is identical to the previously described one: here the larger maximum degree is not an obstacle (see also Theorem 6). Again all auxiliary graphs are weighted regular by the end of the EMBEDDING PHASE with probability at least $2/3$. Moreover, all vertices in L_i have linear degree in $F_i^*(T)$ by the same argument as before (see

Proposition 29 and its proof).

It now remains to show to show that every vertex v in V_i^{Free} has a linear degree in the auxiliary graph $F_i^*(T)$. At this point we deviate from the proof of Theorem 4. Recall that L_i^* was defined as the last λn_i vertices of $X_i \setminus N(S)$ in the arrangeable ordering and $L_i^*(v)$ was defined as the set of vertices $x \in L_i^*$ with $\mathcal{A}_v(y)$ for all $y \in N^-(x)$. We still want to prove that $L_i^*(v)$ is large for every v as this again would imply the linear minimum degree for all $v \in V_i^{Free}$. However, the maximum degree $\Delta(H) \le \xi n/\log n$ does not allow us to partition L_i into sets which are predecessor disjoint of second order any more. This, however, was crucial for our proof of $|L_i^*(v)| \ge 2^{-a^2-1}|L_i^*|$ (see the proof of Lemma 31).

We may, however, alter the definition of the event $\mathcal{A}_v(y)$ to overcome this obstacle. Instead of requiring that $|A(y) \cap N_G(v)|/|A(y)| \ge (2/3)|V_j \cap N_G(v)|/|V_j|$, we now define $\mathcal{A}_v(y)$ in the proof of Theorem 5 to be the event that

$$\frac{|A(y) \cap N(v)|}{|A(y)|} \ge \frac{\iota}{2}\,.$$

We still denote by $L_i^*(v)$ the set of vertices $x \in L_i^*$ with $\mathcal{A}_v(y)$ for all $y \in N^-(x)$. Now the tuple condition guarantees that $|C_{t(y),y} \cap N_G(v)| \ge \iota n_j$ for any $y \in X_j$ and $v \in V \setminus V_j$. Since we chose $V_j^{\mathrm{s}} \subseteq V_j$ randomly we obtain $|C^{\mathrm{s}}_{t(y),y} \cap N_G(v)| \ge (\mu/20)\iota n_j$ for all $y \in X_j$ *almost surely*. The same arguments as in the proof of Proposition 34 imply that $A(y) \approx C^{\mathrm{s}}_{t(y),y}$ for all y that are predecessors of vertices $x \in L_i^*$. Hence,

$$\frac{|A(y) \cap N_G(v)|}{|A(y)|} \approx \frac{|C^{\mathrm{s}}_{t(y),y} \cap N_G(v)|}{|C^{\mathrm{s}}_{t(y),y}|} \approx \frac{(\mu/20)\iota n_j}{(\mu/10)\delta^a n_j} \ge \frac{\iota}{2}$$

for all $x \in L_i^*$ and all $y \in N^-(x)$ almost surely. If this is the case we have $|L_i^*(v)| = |L_i^*| = \lambda n_i$ and therefore the assertion of Lemma 31 holds also in this setting. It remains to show that the same is true for Lemma 32. Indeed, after some appropriate adjustments of the constants, the very same argument implies $\deg_{F_i(T)}(v, L_i) \ge 3\varepsilon' n_i$ for all $i \in [r]$ and $v \in V_i^{Free}$ if $|L_i^*(v)| = |L_i^*|$. More precisely, the change in the definition of $\mathcal{A}_v(y)$ will force smaller values of ε', that is, the constant in the bound of the joint neighbourhood of each $(a+1)$-tuple has to be large compared to the ε in the ε-regularity of the partition $V_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_k$. The constants then relate as

$$0 < \xi \ll \varepsilon \ll \varepsilon' \ll \lambda \ll \gamma \ll \mu, \delta, \iota \le 1\,.$$

The remaining steps in the proof of Theorem 5 are identical to those in the proof of Theorem 4. For $i \in [r]$ the minimum degree in $F_i^*(T)$ together with the weighted regularity implies that $F_i^*(T)$ has a perfect matching. The perfect matching defines a bijection of L_i onto V_i^{Free} that in turn completes the embedding of H into G.

To wrap up, let us quickly comment on the different degree bounds for H in Theorem 4 and Theorem 5. The proof of Theorem 5 just sketched only requires $\Delta(H) = \xi n/\log n$. This is needed to partition L_i^* into *predecessor disjoint sets* in the last step in order to prove the minimum degree for the auxiliary graphs.

Contrary to that the proof of Theorem 4 partitions the vertices of L_i^* into sets which are *predecessor disjoint of second order*, i.e., which do have $N^-(N^-(x)) \cap N^-(N^-(x')) = \emptyset$ for all $x \neq x'$. This is necessary to ensure that there is a linear number of vertices x in L_i^* with $\mathcal{A}_v(y)$ for all $y \in N^-(x)$, i.e., whose predecessors all get embedded into $N_G(v)$ with probability $\delta/3$. More precisely we ensure that, all predecessors y of x have the following property. The predecessors $z_1, \ldots, z_k$ of y are embedded to a k-tuple $(\varphi(z_1), \ldots, \varphi(z_k))$ of vertices in G such that $\bigcap N(\varphi(z_i)) \cap N_G(v) \cap V_{j(y)}$ is large. This fact follows trivially from the tuple condition of Theorem 5 and hence we don't need a partition into predecessor disjoint sets of second order.

5 Optimality

The aim of this section is twofold. Firstly, we shall investigate why the degree bounds given in Theorem 4 and in Theorem 5 are best possible. Secondly, we shall argue why the conditions Theorem 4 imposes on image restrictions are so restrictive.

Optimality of Theorem 5. To argue that the requirement $\Delta(H) \leq n/\log n$ is optimal up to the constant factor we use a construction from [15] and the following proposition.

Proposition 35
For every $\varepsilon > 0$ the domination number of a graph $\mathcal{G}(n, p)$ with high probability is larger than $(1 - \varepsilon)p \log n$.

Proof. The probability that a graph in $\mathcal{G}(n, p)$ has a dominating set of size r is bounded by

$$\binom{n}{r} (1 - (1 - p)^r)^{n-r} \leq \exp\left(r \log n - \exp(-rp)(n - r)\right) .$$

Setting $r = (1 - \varepsilon)p \log n$ we obtain

$$\mathbb{P}\begin{bmatrix}\mathcal{G}(n,p) \text{ has a dominating} \\ \text{set of size } (1-\varepsilon)p\log n\end{bmatrix} \leq \exp\left((1 - \varepsilon)p \log^2 n - \frac{n - (1 - \varepsilon)p \log n}{n^{1-\varepsilon}}\right) \to 0$$

for every (fixed) positive ε. $\square$

Let H be a tree with a root of degree $\frac{1}{2} \log n$, such that each neighbour of this root has $2n/\log n$ leaves as neighbours. This graph H almost surely is not a subgraph of $\mathcal{G}(n, 0.9)$ by Proposition 35 as the neighbours of the root form a dominating set.

Optimality of Theorem 4. The degree bound $\Delta(H) \leq \sqrt{n}/\log n$ is optimal up to the log-factor. More precisely, we can show the following.

Proposition 36
For every $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ there are $n \geq n_0$, an $(\varepsilon, 1/2)$-super-regular pair (V_1, V_2) with $|V_1| = |V_2| = n$ and a tree $T \subseteq K_{n,n}$ with $\Delta(T) \leq \sqrt{n} + 1$, such that (V_1, V_2) does not contain T.

Condition (e) of Theorem 4 allows only a constant number of permissible image restrictions per cluster. The following proposition shows that also this is best possible (up to the value of the constant).

Proposition 37
For every $\varepsilon > 0$, $n_0 \in \mathbb{N}$, and every $w\colon \mathbb{N} \to \mathbb{N}$ which goes to infinity arbitrarily slowly, there are $n \geq n_0$, an $(\varepsilon, 1/2)$-super-regular pair (V_1, V_2) with $|V_1| = |V_2| = n$ and a tree $T \subseteq K_{n,n}$ with $\Delta(T) \leq w(n)$ such that the following is true. The images of $w(n)$ vertices of T can be restricted to sets of size $n/2$ in $V_1 \cup V_2$ such that no embedding of T into (V_1, V_2) respects these image restrictions.

We remark that our construction for Proposition 37 does not require a spanning tree T, but only one on $w(n) + 1$ vertices. Moreover, this proposition shows that the number of admissible image restrictions drops from linear (in the original Blow-up Lemma) to constant (in Theorem 4), if the maximum degree of the target graph H increases from constant to an increasing function.

We now give the constructions that prove these two propositions.

Proof of Proposition 36 (sketch). Let $\varepsilon > 0$ and n_0 be given, choose an integer k such that $1/k \ll \varepsilon$ and an integer n such that $k, n_0 \ll n$, and consider the following bipartite graph $G_k = (V_1 \mathbin{\dot\cup} V_2, E)$ with $|V_1| = |V_2| = n$. Let $W_1, \ldots, W_k$ be a balanced partition of V_1. Now for each *odd* $i \in [k]$ we randomly and independently choose a subset $U_i \subseteq V_2$ of size $n/2$; and we set $U_{i+1} := V_2 \setminus U_i$. Then we insert exactly all those edges into E which have one vertex in W_i and the other in U_i, for $i \in [k]$. Clearly, every vertex in G has degree $n/2$. In addition, using the degree co-degree characterisation of ε-regularity it is not difficult to check that (V_1, V_2) almost surely is ε-regular.

Next, we construct the tree T as follows. We start with a tree T', which consists of a root of degree $\sqrt{n} - 1$ and is such that each child of this root has exactly $\sqrt{n}$ leaves as children. For obtaining T, we then take two copies of T', call their roots x_1 and x_2, respectively, and add an edge between x_1 and x_2. Clearly, the two colour classes of T have size n and $\Delta(T) = \sqrt{n} + 1$.

It remains to show that $T \not\subseteq G_k$. Assume for contradiction that there is an embedding φ of T into G_k such that $\varphi(x_1) \in W_1$. Note that $n - 1$ vertices in T have distance 2 from x_1. Since G_k is bipartite φ has to map these $n - 1$ vertices to V_1. In particular, one of them has to be embedded in W_2. However, the distance between W_1 and W_2 in G_k is greater than 2. $\square$

The proof of Proposition 37 proceeds similarly.

Proof of Proposition 37 (sketch). Let ε, n_0 and w be given, choose n large enough so that $n_0 \leq n$ and $1/w(n) \ll \varepsilon$, and set $k := w(n)$.

We reuse the graph $G_k = (V_1 \mathbin{\dot\cup} V_2, E)$ from the previous proof as ε-regular pair. Now consider any balanced tree T with a vertex x of degree $\Delta(T) = w(n) = k$. Let $\{y_1, \ldots, y_k\}$ be the neighbours of x in T. For $i \in [k]$ we then restrict the image of y_i to $V_2 \setminus U_i$.

We claim that there is no embedding of T into G that respects these image restrictions. Indeed, clearly x has to be embedded into $W_j \subseteq V_1$ for some $j \in [k]$ because its neighbours are image restricted to subsets of V_2. However, by the definition of U_j this prevents y_j from being embedded into $V_2 \setminus U_j$. □

6 Applications

6.1 F-factors for growing degrees

This section is to prove Theorem 7. Our strategy will be to repeatedly embed a collection of copies of F into a super-regular r-tuple in G with the help of the Blow-up Lemma version stated as Theorem 4. The following result by Böttcher, Schacht, and Taraz [4, Lemma 6] says that for $\gamma > 0$ any sufficiently large graph G with $\delta(G) \geq ((r-1)/r+\gamma)|G|$ has a regular partition with a reduced graph R that contains a K_r-factor. Moreover, all pairs of vertices in R that lie in the same K_r span super-regular pairs in G. Let $K_r^{(k)}$ denote the disjoint union of k complete graphs on r vertices each. For all $n, k, r \in \mathbb{N}$, we call an integer partition $(n_{i,j})_{i\in[k],j\in[r]}$ of $[n]$ (with $n_{i,j} \in \mathbb{N}$ for all $i \in [k]$ and $j \in [r]$) r-equitable, if $|n_{i,j} - n_{i,j'}| \leq 1$ for all $i \in [k]$ and $j, j' \in [r]$.

Lemma 38
For all $r \in \mathbb{N}$ and $\gamma > 0$ there exists $\delta > 0$ and $\varepsilon_0 > 0$ such that for every positive $\varepsilon \leq \varepsilon_0$ there exists K_0 and $\xi_0 > 0$ such that for all $n \geq K_0$ and for every graph G on vertex set $[n]$ with $\delta(G) \geq ((r-1)/r + \gamma)n$ there exists $k \in \mathbb{N} \setminus \{0\}$, and a graph $K_r^{(k)}$ on vertex set $[k] \times [r]$ with

(R1) $k \leq K_0$,

(R2) there is an r-equitable integer partition $(m_{i,j})_{i\in[k],j\in[r]}$ of $[n]$ with $(1+\varepsilon)n/(kr) \geq m_{i,j} \geq (1-\varepsilon)n/(kr)$ such that the following holds.[2]

For every integer partition $(n_{i,j})_{i\in[k],j\in[r]}$ of $[n]$ with $m_{i,j} - \xi_0 n \leq n_{i,j} \leq m_{i,j} + \xi_0 n$ there exists a partition $(V_{i,j})_{i\in[k],j\in[r]}$ of V with

(V1) $|V_{i,j}| = n_{i,j}$,

(V2) $(V_{i,j})_{i\in[k],j\in[r]}$ is (ε, δ)-super-regular on $K_r^{(k)}$.

Using this partitioning result for G, Theorem 7 follows easily.

Proof of Theorem 7. We alternatingly choose constants as given by Theorem 4 and Lemma 38. So let $\delta, \varepsilon_0 > 0$ be the constants given by Lemma 38 for r and $\gamma > 0$. Further let $\varepsilon, \alpha > 0$ be the constants given by Theorem 4 for $C = 0$, a, $\Delta_R = r$, $\kappa = 2$, $c = 1$ and δ. We are setting $C = 0$ as we do not use any image restrictions in this proof. If necessary we decrease ε such that $\varepsilon \leq \varepsilon_0$ holds. Let K_0 and $\xi_0 > 0$ be as in Lemma 38 with ε as set before. For r let n_0 be given by Theorem 4. If necessary increase n_0 such that $n_0 \geq K_0$. Finally set $\xi = \xi_0$. In the following we assume that

(i) G is of order $n \geq n_0$ and has $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$, and

[2] The upper bound on $m_{i,j}$ is implicit in the proof of the lemma but not explicitly stated in [4].

(ii) H is an a-arrangeable, r-chromatic F-factor with $|F| \le \xi n$, $\Delta(F) \le \sqrt{n}/\log n$. We need to show that $H \subseteq G$. For this purpose we partition H into subgraphs $H_1, \dots, H_k$, where H_i is to be embedded into $\cup_{j\in[r]} V_{i,j}$ later, as follows. Let $(m_{i,j})_{i\in[k],j\in[r]}$ be an r-equitable integer partition of $[n]$ with $m_{i,j} \ge (1-\varepsilon)n/(kr)$ as given by Lemma 38. For $i = 1, \dots, k-1$ we choose ℓ_i such that both

$$\big|m_{i,j} - \ell_i|F|\big| \le |F|, \qquad \text{and} \qquad \left|\sum_{i'\le i}(m_{i',j} - \ell_{i'}|F|)\right| \le |F| \tag{39}$$

for all $j \in [r]$. Let $n_{i,j} = \ell_i|F|$ for all $j \in [r]$ and set H_i to be $\ell_i r$ copies of F. Note that there exists an r-colouring of H_i in which each colour class $X_{i,j}$ has exactly $n_{i,j}$ vertices. Finally H_k is set to be $H \setminus (H_1 \cup \dots \cup H_{k-1})$. Let $\chi : V(H_k) \to [r]$ be a colouring of H_k where the colour-classes have as equal sizes as possible and set $n_{k,j} := |\chi^{-1}(j)|$ and $X_{k,j} := \chi^{-1}(j)$ for $j \in [r]$. It follows from (39) that

$$|n_{i,j} - m_{i,j}| \le |F| \le \xi_0 n \tag{40}$$

for all $i \in [k]$, $j \in [r]$. Thus there exists a partition $(V_{i,j})_{i\in[k],j\in[r]}$ of $V(G)$ with properties (V1) and (V2) by Lemma 38.

We apply Theorem 4 to embed H_i into $G[V_{i,1} \cup \dots \cup V_{i,r}]$ for every $i \in [k]$. Note that we have partitioned $V(H_i) = X_{i,1} \cup \dots \cup X_{i,r}$ in such a way that $|X_{i,j}| = n_{i,j}$ and $vw \in E(H_i)$ implies $v \in X_{i,j}$, $w \in X_{i,j'}$ with $j \ne j'$. Now properties (V1), (V2) guarantee that $|V_{i,j}| = n_{i,j}$ and $(V_{i,j}, V_{i,j'})$ is an (ε,δ)-super-regular pair in G for all $i \in [k]$ and $j, j' \in [r]$ with $j \ne j'$. It follows that H_i is a subgraph of $G[V_{i,1} \cup \dots \cup V_{i,r}]$ by Theorem 4. □

6.2 Random graphs and universality

Next we prove Theorem 8, which states that $G = \mathcal{G}(n,p)$ is universal for the class of a-arrangeable bounded degree graphs, $\mathcal{H}_{n,a,\xi} = \{H : |H| = n, H \ a\text{-arr.}, \Delta(H) \le \xi n/\log n\}$.

To prove this we will find a balanced partition of G and apply Theorem 5. For this purpose we also have to find a balanced partition of the graphs $H \in \mathcal{H}_{n,a,\xi}$. To this end we shall use the following result of Kostochka, Nakprasit, and Pemmaraju [22]. A graph has a *balanced k-colouring* if the graph has a proper colouring with at most k colours such that the sizes of the colour classes differ by at most 1.

Theorem 39 (Theorem 4 from [22])
Every a-arrangeable[3] graph H with $\Delta(H) \le n/15$ has a balanced k-colouring for each $k \ge 16a$.

Proof of Theorem 8. Let a and p be given. Set $\Delta_R := 16a$, $\kappa = 1$, $\iota := \frac{1}{2}p^{a+1}$, $\delta := p/2$ and let R be a complete graph on $16a$ vertices. Set $r := 16a$ and let ε, ξ, n_0 as given by Theorem 5. Let $n \ge n_0$ and let $V = V_1 \dot\cup \dots \dot\cup V_r$ be a balanced partition of $[n]$.

[3] In fact [22] shows this result for the more general class of a-degenerate graphs.

Then we generate a random graph $G = \mathcal{G}(n,p)$ on vertex set $[n]$. Every pair (V_i, V_j) is $(\varepsilon, p/2)$-super-regular in G with high probability. Furthermore with high probability we have that every tuple $(u_1, \dots, u_{a+1}) \subseteq V \setminus V_i$ satisfies $|\bigcap_{j\in[a+1]} N_G(u_j) \cap V_i| \geq \iota |V_i|$. So assume this is the case and let $H \in \mathcal{H}_{n,a,\xi}$. We partition H into $16a$ equally sized stable sets with the help of Theorem 39. Thus H satisfies the requirements of Theorem 5 and H embeds into G. $\square$

References

[1] P. Allen, J. Skokan, and A. Würfl, *Maximum planar subgraphs in dense graphs.* In preparation.

[2] N. Alon and M. Capalbo, *Sparse universal graphs for bounded-degree graphs.*, Random Struct. Algorithms, 31 (2007), pp. 123–133.

[3] N. Alon and R. Yuster, *H-factors in dense graphs*, J. Combin. Theory Ser. B, 66 (1996), pp. 269–282.

[4] J. Böttcher, M. Schacht, and A. Taraz, *Proof of the bandwidth conjecture of Bollobás and Komlós.*, Mathematische Annalen, 343 (2009), pp. 175–205.

[5] J. Böttcher, A. Taraz, and A. Würfl, *Spanning embeddings of arrangeable graphs with sublinear bandwidth.* In preparation.

[6] G. Chen and R. H. Schelp, *Graphs with linearly bounded ramsey numbers.*, J. Comb. Theory, Ser. B, 57 (1993), pp. 138–149.

[7] V. Chvátal, V. Rödl, E. Szemerédi, and W. T. Trotter, Jr., *The Ramsey number of a graph with bounded maximum degree*, J. Combin. Theory Ser. B, 34 (1983), pp. 239–243.

[8] A. Czygrinow and V. Rödl, *An algorithmic regularity lemma for hypergraphs.*, SIAM J. Comput., 30 (2000), pp. 1041–1066.

[9] A. Hajnal and E. Szemerédi, *Proof of a conjecture of P. Erdős*, in Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 601–623.

[10] H. Huang, C. Lee, and B. Sudakov, *Bandwidth theorem for random graphs*, J. Combin. Theory Ser. B, 102 (2012), pp. 14–37.

[11] S. Janson, T. Łuczak, and A. Ruciński, *Random Graphs*, John Wiley and Sons, 2000.

[12] H. A. Kierstead and W. T. Trotter, *Planar graph coloring with an uncooperative partner*, in Planar graphs (New Brunswick, NJ, 1991), vol. 9 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Amer. Math. Soc., Providence, RI, 1993, pp. 85–93.

[13] J. Komlós, *The Blow-up Lemma*, Combin. Probab. Comput., 8 (1999), pp. 161–176. Recent trends in combinatorics (Mátraháza, 1995).

[14] ——, *Tiling Turán theorems*, Combinatorica, 20 (2000), pp. 203–218.

[15] J. Komlós, G. N. Sárközy, and E. Szemerédi, *Proof of a packing conjecture of Bollobás*, Combinatorics, Probability & Computing, 4 (1995), pp. 241–255.

[16] ——, *On the square of a Hamiltonian cycle in dense graphs*, in Proceedings of the Seventh International Conference on Random Structures and Algorithms (Atlanta, GA, 1995), vol. 9, 1996, pp. 193–211.

[17] ——, *Blow-up Lemma*, Combinatorica, 17 (1997), pp. 109–123.

[18] ——, *An algorithmic version of the Blow-up Lemma*, Random Structures Algorithms, 12 (1998), pp. 297–312.

[19] ——, *On the Pósa-Seymour conjecture*, J. Graph Theory, 29 (1998), pp. 167–176.

[20] ——, *Proof of the Seymour conjecture for large graphs*, Ann. Comb., 2 (1998), pp. 43–60.

[21] ——, *Proof of the Alon-Yuster conjecture*, Discrete Math., 235 (2001), pp. 255–269. Combinatorics (Prague, 1998).

[22] A. V. Kostochka, K. Nakprasit, and S. V. Pemmaraju, *On equitable coloring of d-degenerate graphs*, SIAM J. Discrete Math., 19 (2005), pp. 83–95.

[23] D. Kühn and D. Osthus, *Spanning triangulations in graphs*, J. Graph Theory, 49 (2005), pp. 205–233.

[24] ——, *Embedding large subgraphs into dense graphs*, in Surveys in combinatorics 2009, vol. 365 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2009, pp. 137–167.

[25] ——, *The minimum degree threshold for perfect graph packings*, Combinatorica, 29 (2009), pp. 65–107.

[26] D. Kühn, D. Osthus, and A. Taraz, *Large planar subgraphs in dense graphs*, J. Combin. Theory Ser. B, 95 (2005), pp. 263–282.

[27] O. Riordan, *Spanning subgraphs of random graphs*, Combin. Probab. Comput., 9 (2000), pp. 125–148.

[28] V. Rödl and A. Ruciński, *Perfect matchings in ε-regular graphs and the blow-up lemma*, Combinatorica, 19 (1999), pp. 437–452.

[29] V. Rödl and R. Thomas, *Arrangeability and clique subdivisions*, in The mathematics of Paul Erdős, II, vol. 14 of Algorithms Combin., Springer, Berlin, 1997, pp. 236–239.

Appendix

Weighted regularity

In this section we provide some background on *weighted regularity*. In particular, we supplement the proofs of Lemma 17 and Lemma 18. We start with a short introduction to the results on weighted regularity by Czygrinow and Rödl [8]. Their focus lies on hypergraphs. However, we only present the graph case here.

Czygrinow and Rödl define their weight function on the set of edges (whereas in our scenario we have a bipartite graph with weights on the vertices of one class). They consider weighted graphs $G = (V, \widetilde{\omega})$ where $\widetilde{\omega} : V \times V \to \mathbb{N}_{\geq 0}$. One can think of $\widetilde{\omega}(x, y)$ as the multiplicity of the edge (x, y). Their weighted degree and co-degree for $x, y \in V$ are then defined as

$$\deg^*_{\widetilde{\omega}}(x) := \sum_{y \in V} \widetilde{\omega}(x, y), \qquad \deg^*_{\widetilde{\omega}}(x, y) := \sum_{z \in V} \widetilde{\omega}(x, z)\widetilde{\omega}(y, z)\,.$$

For disjoint $A, B \subseteq V$ they define

$$d^*_{\widetilde{\omega}}(A, B) = \frac{\sum \widetilde{\omega}(x, y)}{K|A|\,|B|}\,,$$

where the sum is over all pairs $(x, y) \in A \times B$ and $K := 1 + \max\{\widetilde{\omega}(x, y) : (x, y) \in V \times V\}$.[4] A pair (A, B) in $G = (V, \widetilde{\omega})$ with $A \cap B = \emptyset$ is called $(\varepsilon, \widetilde{\omega})$*-regular* if

$$|d^*_{\widetilde{\omega}}(A, B) - d^*_{\widetilde{\omega}}(A', B')| < \varepsilon$$

for all $A' \subseteq A$ with $|A'| \geq \varepsilon|A|$ and all $B' \subseteq B$ with $|B'| \geq \varepsilon|B|$. As in the unweighted case, regular pairs can be characterised by the degree and co-degree distribution of their vertices. The following lemma (see [8, Lemma 4.2]) shows that a pair is weighted regular in the setting of Czygrinow and Rödl if most of the vertices have the correct weighted degree and most of the pairs have the correct weighted co-degree.

Lemma 40 (Czygrinow, Rödl [8])
Let $G = (A \cupdot B, \widetilde{\omega})$ be a weighted graph with $|A| = |B| = n$ and let $\varepsilon, \xi \in (0, 1)$, $\xi^2 < \varepsilon$, $n \geq 1/\xi$. Assume that both of the following conditions are satisfied:

(i') $|\{x \in A : |\deg^*_{\widetilde{\omega}}(x) - K\, d^*_{\widetilde{\omega}}(A, B)n| > K\xi^2 n\}| < \xi^2 n$, and

(ii') $\left|\left\{\{x_i, x_j\} \in \binom{A}{2} : |\deg^*_{\widetilde{\omega}}(x_i, x_j) - K^2 d^*_{\widetilde{\omega}}(A, B)^2 n| \geq K^2 \xi n\right\}\right| \leq \xi\binom{n}{2}$.

Then for every $A' \subseteq A$ with $|A'| \geq \varepsilon n$ and every $B' \subseteq B$ with $|B'| \geq \varepsilon n$ we have

$$|d^*_{\widetilde{\omega}}(A', B') - d^*_{\widetilde{\omega}}(A, B)| \leq 2\frac{\xi^2}{\varepsilon} + \frac{\sqrt{5\xi}}{\varepsilon^2 - \varepsilon\xi^2}\,.$$

The assertion of Lemma 40 implies that the pair (A, B) is $(\varepsilon', \widetilde{\omega})$-regular, where $\varepsilon' = \max\{\varepsilon, 2\xi^2/\varepsilon + \sqrt{5\xi}/(\varepsilon^2 - \varepsilon\xi^2)\}$, if the conditions of the lemma are satisfied.

[4] Czygrinow and Rödl require K to be strictly larger than the maximal weight for technical reasons.

Our goal is to translate this result into our setting of weighted regularity (see Section 2.2). We shortly recall our definition of weighted graphs and weighted regularity before we restate and prove Lemma 17.

Let $G = (A \cupdot B, E)$ be a bipartite graph and $\omega : A \to [0,1]$ be our weight function for G. We define the weighted degree of a vertex $x \in A$ to be $\deg_\omega(x) = \omega(x)|N(x,B)|$ and the weighted co-degree of $x, y \in A$ as $\deg_\omega(x,y) = \omega(x)\omega(y)|N(x,B) \cap N(y,B)|$. Similarly, the weighted density of a pair (A', B') is defined as

$$d_\omega(A', B') := \sum_{x \in A'} \frac{\omega(x)|N(x,B')|}{|A'| \cdot |B'|} .$$

Again the pair (A, B) is called weighted ε-regular if

$$|d_\omega(A,B) - d_\omega(A', B')| \le \varepsilon$$

for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge \varepsilon|A|$ and $|B'| \ge \varepsilon|B|$. We now prove Lemma 17, which we restate here for the reader's convenience.

Lemma (Lemma 17)
Let $\varepsilon > 0$ and $n \ge \varepsilon^{-6}$. Further let $G = (A \cupdot B, E)$ be a bipartite graph with $|A| = |B| = n$ and let $\omega : A \to [\varepsilon, 1]$ be a weight function for G. If

(i) $|\{x \in A : |\deg_\omega(x) - d_\omega(A,B)n| > \varepsilon^{14}n\}| < \varepsilon^{12}n$ and

(ii) $|\{\{x,y\} \in \binom{A}{2} : |\deg_\omega(x,y) - d_\omega(A,B)^2 n| \ge \varepsilon^9 n\}| \le \varepsilon^6 \binom{n}{2}$

then (A, B) is a weighted 3ε-regular pair.

Proof of Lemma 17. Let $\varepsilon > 0$, $G = (A \cup B, E)$ and $\omega : A \to [\varepsilon, 1]$ satisfy the requirements of the lemma. From this ω we define a weight function $\widetilde{\omega} : A \times B \to \mathbb{N}_{\ge 0}$ in the setting of Lemma 40. For $(x, y) \in A \times B$ we set

$$\widetilde{\omega}(x,y) := \begin{cases} \lceil C \cdot \omega(x) \rceil & \text{if } \{x,y\} \in E, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon^{-13} - 1 \le C \le \varepsilon^{-14}$ is chosen such that $K = \max\{\widetilde{\omega}(x,y) + 1 : (x,y) \in A \times B\} \ge \varepsilon^{-13}$. (This is possible unless $E = \emptyset$.) Note that our choice of constants implies $K/C \le 1 + 2\varepsilon^{13}$. Moreover, let $d^*_{\widetilde{\omega}}(A,B)$ be defined as above. The definition of $\widetilde{\omega}$ implies

$$C \deg_\omega(x) \le \deg^*_{\widetilde{\omega}}(x) \le C \deg_\omega(x) + |N(x,B)| \text{ and} \tag{41}$$
$$C^2 \deg_\omega(x,y) \le \deg^*_{\widetilde{\omega}}(x,y) \le C^2 \deg_\omega(x,y) + (2C+1)|N(x,B) \cap N(y,B)| \tag{42}$$

for all $x, y \in A$. Here the second inequality follows from

$$\lceil C \cdot \omega(x) \rceil^2 \le (C \cdot \omega(x) + 1)^2 \le C^2 \big(\omega(x)\big)^2 + 2C + 1 .$$

Moreover,

$$C d_\omega(A', B') \le K d^*_{\widetilde{\omega}}(A', B') \le C d_\omega(A', B') + 1 \tag{43}$$

for all $A' \subseteq A$, $B' \subseteq B$ which in turn implies that

$$\big(C\, d_\omega(A', B')\big)^2 - \big(K\, d^*_{\widetilde{\omega}}(A', B')\big)^2 \leq 1 \cdot (C + K) \tag{44}$$

for all $A' \subseteq A$, $B' \subseteq B$.

We now verify that conditions (i) and (ii) of Lemma 17 imply conditions (i') and (ii') of Lemma 40. Set $\xi := \varepsilon^6$ and let $x \in A$ be such that $|\deg_\omega(x) - d_\omega(A, B)n| \leq \varepsilon^{14} n$. It follows from (41), (43) and the triangle inequality that

$$\begin{aligned} |\deg^*_{\widetilde{\omega}}(x) - K\, d^*_{\widetilde{\omega}}(A, B)n| &\leq |\deg^*_{\widetilde{\omega}}(x) - C \deg_\omega(x)| \\ &\quad + |C\, \deg_\omega(x) - C\, d_\omega(A, B)n| \\ &\quad + |C\, d_\omega(A, B)n - K\, d^*_{\widetilde{\omega}}(A, B)n| \\ &\leq n + C\varepsilon^{14} n + n \leq 3n \\ &\leq K\xi^2 n\,. \end{aligned}$$

Hence, condition (i) implies condition (i').

Now let $\{x, y\} \in \binom{A}{2}$ satisfy $|\deg_\omega(x, y) - d_\omega(A, B)^2 n| < \varepsilon^9 n$. It follows from (42), (43) and (44) that

$$\begin{aligned} \left|\deg^*_{\widetilde{\omega}}(x, y) - K^2 d^*_{\widetilde{\omega}}(A, B)^2 n\right| &\leq |\deg^*_{\widetilde{\omega}}(x, y) - C^2 \deg_\omega(x, y)| \\ &\quad + |C^2 \deg_\omega(x, y) - C^2 d_\omega(A, B)^2 n| \\ &\quad + \left|C^2 d_\omega(A, B)^2 n - K^2 d^*_{\widetilde{\omega}}(A, B)^2 n\right| \\ &< (2C + 1)n + C^2 \varepsilon^9 n + (C + K)n \\ &\leq K^2 \xi n\,, \end{aligned}$$

where the last inequality is due to $C \leq K/\varepsilon$. Thus, condition (ii) implies condition (ii').

We conclude that $G = (A \mathbin{\dot\cup} B, \widetilde{\omega})$ satisfies the requirements of Lemma 40. Hence every $A' \subseteq A$ with $|A'| \geq \varepsilon n$ and every $B' \subseteq B$ with $|B'| \geq \varepsilon n$ has

$$|d^*_{\widetilde{\omega}}(A', B') - d^*_{\widetilde{\omega}}(A, B)| \leq 2\frac{\xi^2}{\varepsilon} + \frac{\sqrt{5\xi}}{\varepsilon^2 - \varepsilon\xi^2} \leq \frac{5}{2}\varepsilon\,.$$

Together with (43) and the fact that $K/C \leq 1 + 2\varepsilon^{13}$ this finishes the proof as we have

$$\begin{aligned} |d_\omega(A', B') - d_\omega(A, B)| &\leq |d_\omega(A', B') - \tfrac{K}{C} d^*_{\widetilde{\omega}}(A', B')| \\ &\quad + |\tfrac{K}{C} d^*_{\widetilde{\omega}}(A', B') - \tfrac{K}{C} d^*_{\widetilde{\omega}}(A, B)| \\ &\quad + |\tfrac{K}{C} d^*_{\widetilde{\omega}}(A, B) - d_\omega(A, B)| \\ &\leq \tfrac{1}{C} + \tfrac{K}{C}\tfrac{5}{2}\varepsilon + \tfrac{1}{C} \\ &\leq 3\varepsilon\,. \end{aligned}$$

□

We want to point out that the requirement that ω is at least ε does not cause any problem when we apply Lemma 17 because one could simply increase the weight of all vertices x with $\omega(x) < \varepsilon$ to ε without changing the weighted densities in the subpairs by more than ε. Hence a graph with an arbitrary weight function is weighted 2ε-regular if the graph with the modified weight function is weighted ε-regular.

The remainder of this section is dedicated to the proof of Lemma 18 which we restate here.

Lemma (Lemma 18)
Let $\varepsilon > 0$ and let $G = (A \cup B, E)$ with $|V_i| = n$ and weight function $\omega : A \to [\sqrt{\varepsilon}, 1]$ be a weighted ε-regular pair. If $\deg(x) > 2\sqrt{\varepsilon}n$ for all $x \in A \cup B$ then G contains a perfect matching.

Proof of Lemma 18. In order to prove that $G = (A \cup B, E)$ has a perfect matching, we will verify the König–Hall criterion for G, i.e., we will show that $|N(S)| \geq |S|$ for every $S \subseteq A$. We distinguish three cases.

Case 1, $|S| < \varepsilon n$: The minimum degree $\deg(x) \geq 2\sqrt{\varepsilon}n$ implies $|N(S)| \geq 2\sqrt{\varepsilon}n \geq |S|$ for any non-empty set S.

Case 2, $\varepsilon n \leq |S| \leq (1-\varepsilon)n$: Note that $\deg(x) > 2\sqrt{\varepsilon}n$ and $\omega(x) \geq \sqrt{\varepsilon}$ for all $x \in A$ implies that $d_\omega(A, B) > 2\varepsilon$. We now set $T = B \setminus N(S)$. Since $d_\omega(S, T) = 0$ and (A, B) is a weighted-ε-regular pair with weighted density greater than 2ε we conclude that $|T| < \varepsilon n$.

Case 3, $|S| > (1-\varepsilon)n$: For every $y \in B$ we have $|S| + |N(y)| \geq (1 - \varepsilon + 2\sqrt{\varepsilon})n > n$ and thus $N(y) \cap S \neq \emptyset$. It follows that $N(S) = B$ if $|S| > (1-\varepsilon)n$. □

Chernoff type bounds

The analysis of our randomised greedy embedding (see Section 3.2) repeatedly uses concentration results for random variables. Those random variables are the sum of Bernoulli variables. If these are mutually independent we use a Chernoff bound (see, e.g., [11, Corollary 2.3]).

Theorem 41 (Chernoff bound)
Let $\mathcal{A} = \sum_{i\in[n]} \mathcal{A}_i$ be a binomially distributed random variable with $\mathbb{P}[\mathcal{A}_i] = p$ for all $i \in [n]$. Further let $c \in [0, 3/2]$. Then

$$\mathbb{P}[|\mathcal{A} - pn| \geq c \cdot pn] \leq \exp\left(-\frac{c^2}{3}pn\right).$$

However, we also consider scenarios where the Bernoulli variables are not independent.

Lemma (Lemma 19)
Let $0 \leq p_1 \leq p_2 \leq 1$, $0 < c \leq 1$. Further let $\mathcal{A}_i$ for $i \in [n]$ be a 0-1-random variable and set $\mathcal{A} := \sum_{i\in[n]} \mathcal{A}_i$. If

$$p_1 \leq \mathbb{P}\left[\mathcal{A}_i = 1 \;\middle|\; \begin{array}{l} \mathcal{A}_j = 1 \text{ for all } j \in J \text{ and} \\ \mathcal{A}_j = 0 \text{ for all } j \in [i-1] \setminus J \end{array}\right] \leq p_2 \tag{45}$$

for every $i \in [n]$ and every $J \subseteq [i-1]$ then

$$\mathbb{P}[\mathcal{A} \leq (1-c)p_1 n] \leq \exp\left(-\frac{c^2}{3}p_1 n\right)$$

and

$$\mathbb{P}[\mathcal{A} \geq (1+c)p_2 n] \leq \exp\left(-\frac{c^2}{3}p_2 n\right).$$

The somewhat technical conditioning in (45) allows us to bound the probability for the event $\mathcal{A}_i = 1$ even if we condition on any outcome of the events $\mathcal{A}_j$ with $j < i$.

The idea of the proof now is to relate the random variable $\mathcal{A}$ to a truly binomially distributed random variable and then use a Chernoff bound.

Proof of Lemma 19. For $k, \ell \in \mathbb{N}_0$ define $a_{\ell,k} = \mathbb{P}[\sum_{i\leq \ell} \mathcal{A}_i \leq k]$ and $b_{\ell,k} = \mathbb{P}[B_{\ell,p_1} \leq k]$ where B_{ℓ,p_1} is a binomially distributed random variable with parameters ℓ and p_1. So both $a_{\ell,k}$ and $b_{\ell,k}$ give a probability that a random variable (depending on ℓ and p_1) is below a certain value k. The following claim relates these two probabilities.

Claim 42 *For every $k \geq 0$, $\ell \geq 0$ we have $a_{\ell,k} \leq b_{\ell,k}$.*

Proof. We will prove the claim by induction on ℓ. For $\ell = 0$ we trivially have $a_{0,k} = 1 = b_{0,k}$ for all $k \geq 0$. Now assume that the claim is true for $\ell - 1$ and every $k \geq 0$. Now

$$a_{\ell,0} \leq (1-p_1)a_{\ell-1,0} \leq (1-p_1)b_{\ell-1,0} = b_{\ell,0}.$$

As

$$\mathbb{P}\left[\mathcal{A}_\ell = 1 \;\middle|\; \begin{array}{l}\mathcal{A}_j = 1 \text{ for all } j \in J \text{ and}\\ \mathcal{A}_j = 0 \text{ for all } j \in [\ell-1]\setminus J\end{array}\right] \geq p_1$$

for every $J \subseteq [\ell-1]$ it follows that for $k \geq 1$

$$a_{\ell,k} \leq a_{\ell-1,k-1} + (a_{\ell-1,k} - a_{\ell-1,k-1})(1-p_1). \tag{46}$$

This upper bound on $a_{\ell,k}$ implies that for every $k \geq 1$ we have

$$\begin{aligned} a_{\ell,k} &\overset{(46)}{\leq} p_1 \cdot a_{\ell-1,k-1} + (1-p_1)\cdot a_{\ell-1,k} \\ &\leq p_1 \cdot b_{\ell-1,k-1} + (1-p_1)\cdot b_{\ell-1,k} \\ &= p_1\,\mathbb{P}[B_{\ell-1,p_1} \leq k-1] + (1-p_1)\,\mathbb{P}[B_{\ell-1,p_1} \leq k] \\ &= \mathbb{P}[B_{\ell-1,p_1} \leq k-1] + (1-p_1)\,\mathbb{P}[B_{\ell-1,p_1} = k] \\ &= \mathbb{P}[B_{\ell,p_1} \leq k] = b_{\ell,k}. \end{aligned}$$

Here the second inequality is due to the induction hypothesis. This finishes the induction step and the proof of the claim. $\square$

Now the first inequality of the lemma follows immediately. We set $\ell = n$, $k = (1-c)p_1 n$ and obtain

$$\mathbb{P}[\mathcal{A} \leq (1-c)p_1 n] = a_{n,(1-c)p_1 n} \leq b_{n,(1-c)p_1 n} = \mathbb{P}[B_{n,p_1} \leq (1-c)p_1 n] \leq \exp\left(-\frac{c^2}{3}p_1 n\right),$$

where the last inequality follows by Theorem 41.

The second assertion of the lemma follows by an analogous argument: set $a_{\ell,k} = \mathbb{P}[\sum_{i\leq \ell} \mathcal{A}_i \geq k]$ and $b_{\ell,k} = \mathbb{P}[B_{\ell,p_2} \geq k]$ and obtain

$$a_{\ell,k} \leq a_{\ell-1,k} + (a_{\ell-1,k-1} - a_{\ell-1,k})p_2 \,. \tag{47}$$

It follows by induction on ℓ that

$$\begin{aligned} a_{\ell,k} &\overset{(47)}{\leq} (1-p_2)\cdot a_{\ell-1,k} + p_2 \cdot a_{\ell-1,k-1} \\ &\leq (1-p_2)\cdot b_{\ell-1,k} + p_2 \cdot b_{\ell-1,k-1} \\ &= \mathbb{P}[B_{\ell-1,p_2} \geq k] + \mathbb{P}[B_{\ell-1,p_2} = k-1]p_2 \\ &= \mathbb{P}[B_{\ell,p_2} \geq k] = b_{\ell,k}\,. \end{aligned}$$

Once more the second inequality follows from the induction hypothesis. Setting $\ell = n$ and $k = (1+c)p_2 n$ and using Theorem 41 again then finishes the proof. □

In addition we need a similar result with a more complex setup.

Lemma (Lemma 20)
Let $0 < p$ and $a, m, n \in \mathbb{N}$. Further let $\mathcal{I} \subseteq \mathcal{P}([n]) \setminus \{\emptyset\}$ be a collection of m disjoint sets with at most a elements each. For every $i \in [n]$ let $\mathcal{A}_i$ be a 0-1-random variable. Further assume that for every $I \in \mathcal{I}$ and every $k \in I$ we have

$$\mathbb{P}\left[\mathcal{A}_k = 1 \;\middle|\; \begin{array}{l} \mathcal{A}_j = 1 \textit{ for all } j \in J \textit{ and} \\ \mathcal{A}_j = 0 \textit{ for all } j \in [k-1]\setminus J \end{array}\right] \geq p$$

for every $J \subseteq [k-1]$ with $[k-1] \cap I \subseteq J$. Then

$$\mathbb{P}\left[\left|\{I \in \mathcal{I} : \mathcal{A}_i = 1 \textit{ for all } i \in I\}\right| \geq \tfrac{1}{2}p^a m\right] \geq 1 - 2\exp\left(-\frac{1}{12}p^a m\right).$$

Proof of Lemma 20. Let $p > 0$, $a, m, n \in \mathbb{N}$ and $\mathcal{I}$ be given. We order the elements of $\mathcal{I}$ as $\mathcal{I} = \{I_1, \ldots, I_m\}$ by their respective largest index. This means, the I_j are sorted such that $j' < j$ implies that there is an index $i_j \in I_j$ with $i < i_j$ for all $i \in I_{j'}$. For $i \in [m]$ we now define events $\mathcal{B}_i$ as

$$\mathcal{B}_i := \begin{cases} 1 & \text{if } \mathcal{A}_j = 1 \text{ for all } j \in I_i, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the events $\mathcal{B}_i$ satisfy equation (45) where the probability is bounded from below by p^a.

Claim 43 *For every $i \in [m]$ and $J \subseteq [i-1]$ we have*

$$\mathbb{P}\left[\mathcal{B}_i = 1 \;\middle|\; \begin{array}{l} \mathcal{B}_j = 1 \text{ for all } j \in J \text{ and} \\ \mathcal{B}_j = 0 \text{ for all } j \in [i-1] \setminus J \end{array}\right] \geq p^a \,.$$

Proof. Let $i \in [m]$ and $J \subseteq [i-1]$ be given. We assume that $|I| = a$ for ease of notation. (The proof is just the same if $|I| < a$.) So let $I_i = \{i_1, \ldots, i_a\}$ be in ascending order and define $i_0 := 0$. For $v \in \{0,1\}^{i_k - i_{k-1} - 1}$ let $H_k(v)$ be the 0-1-random variable with

$$H_k(v) = \begin{cases} 1 & \mathcal{A}_{i_{k-1}+\ell} = v_\ell \text{ for all } \ell \in [i_k - i_{k-1} - 1], \\ 0 & \text{otherwise.} \end{cases}$$

The rationale for this definition is the following. The outcome of $\mathcal{B}_i$ is determined by the outcome of the random variables $\mathcal{A}_{i_j}$. However, we cannot neglect the random variables $\mathcal{A}_\ell$ for $\ell \notin I_i$ as the $\mathcal{A}_\ell$ are not mutually independent. Instead we condition the probability of $\mathcal{A}_{i_j} = 1$ on possible outcomes of $\mathcal{A}_\ell$ with $\ell < i_j$. Now $H_k(v) = 1$ with $v \in \{0,1\}^{i_k - i_{k-1} - 1}$ represents one outcome for the $\mathcal{A}_\ell$ with $i_{k-1} < \ell < i_k$. We call the $v \in \{0,1\}^{i_k - i_{k-1} - 1}$ the *history* between $\mathcal{A}_{i_{k-1}}$ and $\mathcal{A}_{i_k}$. It follows from the requirements of Lemma 20 that for any tuple $(v_1, \ldots, v_k) \in \{0,1\}^{i_1 - 1} \times \{0,1\}^{i_2 - i_1 - 1} \times \cdots \times \{0,1\}^{i_k - i_{k-1} - 1}$ we have

$$\mathbb{P}\left[\mathcal{A}_{i_k} = 1 \;\middle|\; \begin{array}{l} \mathcal{A}_{i_j} = 1 \text{ for all } j \in [k-1] \text{ and} \\ H_j(v_j) = 1 \text{ for all } j \in [k-1] \end{array}\right] \geq p \,. \tag{48}$$

However, we are not interested in every possible history $(v_1, \ldots, v_k)$ as some of the histories cannot occur simultaneously with the event $\mathcal{B} = 1$ where

$$\mathcal{B} = 1 \text{ if and only if } \begin{bmatrix} \mathcal{B}_j = 1 \text{ for all } j \in J \text{ and} \\ \mathcal{B}_j = 0 \text{ for all } j \in [i-1] \setminus J \end{bmatrix} .$$

For ease of notation we define the following shortcuts

$$\begin{aligned} \mathcal{H}(v_1, \ldots, v_k) = 1 &\text{ if and only if } H_j(v_j) = 1 \text{ for all } j \in [k], \\ \mathcal{A}^{(k)} = 1 &\text{ if and only if } \mathcal{A}_{i_j} = 1 \text{ for all } j \in [k]. \end{aligned}$$

Moreover, we define C_k to be the set of all tuples $(v_1, \ldots, v_k) \in \{0,1\}^{i_1 - 1} \times \{0,1\}^{i_2 - i_1 - 1} \times \cdots \times \{0,1\}^{i_k - i_{k-1} - 1}$ with

$$\mathbb{P}\left[\mathcal{H}(v_1, \ldots, v_k) = 1 \text{ and } \mathcal{B} = 1\right] > 0 \,.$$

In other words, the elements of C_k are those histories that are compatible with the event that we condition on in the claim. Note in particular that $\mathcal{B} = 1$ if and only if there is a $(v_1, \ldots, v_a) \in C_a$ with $\mathcal{H}(v_1, \ldots, v_a) = 1$. With these definitions we can rewrite the probability in the assertion of our claim as

$$\mathbb{P}\left[\mathcal{B}_i = 1 \mid \mathcal{B} = 1\right] = \sum_{(v_1, \ldots, v_a) \in C_a} \mathbb{P}\left[\begin{array}{l} \mathcal{A}^{(a)} = 1 \text{ and} \\ \mathcal{H}(v_1, \ldots, v_a) = 1 \end{array} \;\middle|\; \mathcal{B} = 1\right] .$$

We now prove by induction on k that

$$P_k := \sum_{(v_1,\dots,v_k)\in C_k} \mathbb{P}\left[\begin{matrix}\mathcal{A}^{(k)} = 1 \text{ and} \\ \mathcal{H}(v_1,\dots,v_k) = 1\end{matrix} \;\middle|\; \mathcal{B} = 1\right] \geq p^k \tag{49}$$

for all $k \in [a]$. The induction base $k = 1$ is immediate from the requirements of the lemma as

$$P_1 = \sum_{v_1\in C_1} \mathbb{P}\left[\begin{matrix}\mathcal{A}^{(1)} = 1 \text{ and} \\ \mathcal{H}(v_1) = 1\end{matrix} \;\middle|\; \mathcal{B} = 1\right] \overset{(48)}{\geq} p \sum_{v_1\in C_1} \mathbb{P}\left[\mathcal{H}(v_1) = 1 \;\middle|\; \mathcal{B} = 1\right] = p\,.$$

The last equality above follows by total probability from the definition of C_1. So assume that the induction hypothesis holds for $k-1$. Then

$$\begin{aligned}
P_k &= \sum_{(v_1,\dots,v_k)\in C_k} \mathbb{P}\left[\begin{matrix}\mathcal{A}^{(k)} = 1 \text{ and} \\ \mathcal{H}(v_1,\dots,v_k) = 1\end{matrix} \;\middle|\; \mathcal{B} = 1\right] \\
&= \sum_{(v_1,\dots,v_k)\in C_k} \mathbb{P}\left[\mathcal{A}_{i_k} = 1 \middle| \begin{matrix}\mathcal{B} = 1 \text{ and } \mathcal{A}^{(k-1)} = 1 \text{ and} \\ \mathcal{H}(v_1,\dots,v_k) = 1\end{matrix}\right] \cdot \mathbb{P}\left[\begin{matrix}\mathcal{A}^{(k-1)} = 1 \text{ and} \\ \mathcal{H}(v_1,\dots,v_k) = 1\end{matrix} \middle| \mathcal{B} = 1\right] \\
&\overset{(48)}{\geq} p \cdot \sum_{(v_1,\dots,v_k)\in C_k} \mathbb{P}\left[\begin{matrix}\mathcal{A}^{(k-1)} = 1 \text{ and} \\ \mathcal{H}(v_1,\dots,v_k) = 1\end{matrix} \middle| \mathcal{B} = 1\right] \\
&= p \cdot \sum_{(v_1,\dots,v_{k-1})\in C_{k-1}} \mathbb{P}\left[\begin{matrix}\mathcal{A}^{(k-1)} = 1 \text{ and} \\ \mathcal{H}(v_1,\dots,v_{k-1}) = 1\end{matrix} \middle| \mathcal{B} = 1\right] \\
&= p \cdot P_{k-1} \geq p^k\,.
\end{aligned}$$

The claim now follows as

$$\mathbb{P}[\mathcal{B}_i = 1 \mid \mathcal{B} = 1] = \sum_{(v_1,\dots,v_a)\in C_a} \mathbb{P}\left[\begin{matrix}\mathcal{A}^{(a)} = 1 \text{ and} \\ \mathcal{H}(v_1,\dots,v_a) = 1\end{matrix} \;\middle|\; \mathcal{B} = 1\right] \geq p^a\,.$$

□

We have seen that the $\mathcal{B}_i$ are pseudo-independent and that they have probability at least p^a each. Thus we can apply Lemma 19 and derive

$$\mathbb{P}\left[|\{i \in [m] : \mathcal{B}_i = 1\}| \geq \frac{1}{2}p^a m\right] \geq 1 - 2\exp\left(-\frac{1}{12}p^a m\right)\,.$$

□