

Linear sized homogenous sets

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(joint work with Julia Böttcher & Anusch Taraz)

In the beginning there was...

Conjecture

ERDŐS, HAJNAL 1989

For every graph H there is $\varepsilon(H) > 0$ such that all graphs $G \in \text{FORB}_n^*(H)$ have $\text{hom}(G) \geq n^{\varepsilon(H)}$.

Definition

For (simple, undirected) graphs G, H let

- $\text{hom}(G) = \max\{\alpha(G), \omega(G)\}$,
- $\text{FORB}_n^*(H) = \{G : |G| = n, H \not\preceq G\}$.

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Ramsey: $\text{hom}(G) = \Theta(\log n)$ for arbitrary G

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Theorem

LOEBL, REED, SCOTT, THOMASON, THOMASSÉ, 2010

For every graph H there is $\varepsilon(H) > 0$ such that **almost all** graphs $G \in \text{FORB}_n^*(H)$ have $\text{hom}(G) \geq n^{\varepsilon(H)}$.

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For every graph H there is $\varepsilon(H) > 0$ such that **almost all** graphs $G \in \text{FORB}_n^*(H)$ have $\text{hom}(G) \geq n^{\varepsilon(H)}$.

Question

LOEBL, REED, SCOTT, THOMASON, THOMASSÉ, 2010

Which graphs H have *the property* that there is $\eta(H) > 0$ such that almost all graphs $G \in \text{FORB}_n^*(H)$ have $\text{hom}(G) \geq \eta(H) \cdot n$, i.e.,

$$\frac{|\text{FORB}_{n,\eta(H)}^*(H)|}{|\text{FORB}_n^*(H)|} \rightarrow 0 \quad (n \rightarrow \infty) ?$$

$$\text{FORB}_{n,\eta}^*(H) = \{G \in \text{FORB}_n^*(H) : \text{hom}(G) < \eta n\}$$

What is known

An easy case

For $H = P_3$ every graph in $\text{FORB}_n^*(H)$ is the disjoint union of cliques.

- $\text{FORB}_n^*(H)$ is dominated by graphs G with $\omega(G) = \Theta(\log n)$
- $\Rightarrow P_3$ does **not** have *the property*

What is known

An easy case

For $H = P_3$ every graph in $\text{FORB}_n^*(H)$ is the disjoint union of cliques.

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- $\Rightarrow P_3$ does **not** have *the property*

Loebl, Reed, Scott, Thomason, Thomassé:

- P_4 might not have the property
- all other graphs seem to have *the property*

The pentagon

$H = C_5$ has the property even in a **stricter sense**:

Theorem

BÖTTCHER, TARAZ, W, 2010

For every $c \in (0, 1)$ there exists $\eta > 0$ such that

$$\frac{|\text{FORB}_{n,\eta}^*(C_5, c)|}{|\text{FORB}_n^*(C_5, c)|} \rightarrow 0 \quad (n \rightarrow \infty).$$

$$\text{FORB}_n^*(H, c) = \{G : H \not\subseteq G, |G| = n, \|G\| = c \binom{n}{2}\}$$

Sketch of the proof

Theorem

For every $c \in (0, 1)$ there exists $\eta > 0$ such that

$$\frac{|\text{FORB}_{n,\eta}^*(C_5, c)|}{|\text{FORB}_n^*(C_5, c)|} \rightarrow 0 \quad (n \rightarrow \infty).$$

1. find a lower bound on the size of $|\text{FORB}_n^*(C_5, c)|$
2. apply the regularity lemma to all graphs in $\text{FORB}_n^*(C_5, c)$
3. count graphs that share a reduced graph
4. have a closer look at regular pairs
5. count graphs that share a reduced graph **again**

Step 1:

Define

$$h(c) := \begin{cases} H(2c)/2 & \text{if } 0 < c < \frac{1}{4}, \\ 1/2 & \text{if } \frac{1}{4} \leq c \leq \frac{3}{4}, \\ H(2c-1)/2 & \text{otherwise.} \end{cases}$$

Here $H(c) = -c \log c - (1-c) \log(1-c)$ is the binary entropy function.

Binomial coefficients

For every $c \in (0, 1)$

$$\binom{n}{cn} = 2^{H(c)n + o(n)}.$$

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Lemma

For every $c \in (0, 1)$

$$|\text{FORB}_n^*(C_5, c)| \geq 2^{h(c) \binom{n}{2} + o(1) \binom{n}{2}}.$$

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Our Aim:

Lemma

For every $c \in (0, 1)$ there is $\eta > 0$ such that

$$|\text{FORB}_{n,\eta}^*(C_5, c)| \leq 2^{(h(c) - r(c))\binom{n}{2} + o(1)\binom{n}{2}},$$

where $r : (0, 1) \rightarrow \mathbb{R}^+$.

Step 2:

Regularize all graphs in $\text{FORB}_n^*(C_5, c)$ to obtain *coloured reduced graphs*:

Definition

A graph G with regular partition $V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_k$ has the **coloured reduced graph** $R = (V_R, E_R, \sigma)$ if

$$\sigma : \quad E_R \rightarrow \{\blacksquare, \blacksquare, \square\}$$

$$\{v_i, v_j\} \mapsto \begin{cases} \blacksquare & \text{if } d(V_i, V_j) > (1 - d) \\ \blacksquare & \text{if } d(V_i, V_j) \in [d, 1 - d] \\ \square & \text{if } d(V_i, V_j) < d. \end{cases}$$

Step 3:

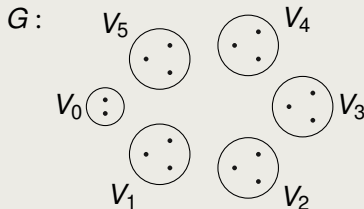
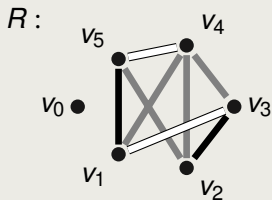
Let $\mathcal{R}(R, \varepsilon, n, c) = \{G \in \text{FORB}_n^*(C_5, c) : G \text{ has col. red. graph } R\}$.

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Remark

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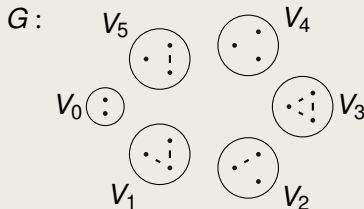
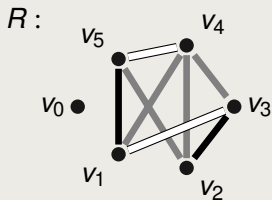
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- of choices for edges inside clusters,



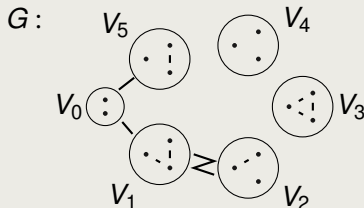
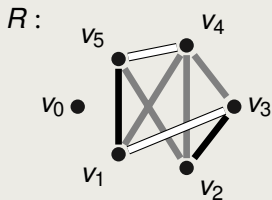
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Remark

An upper bound on $|\mathcal{R}(R, \varepsilon, n, c)|$ is given by the product of the number

- of choices for edges inside clusters,
- of choices for edges in irregular pairs and to V_0 ,



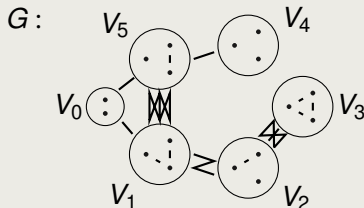
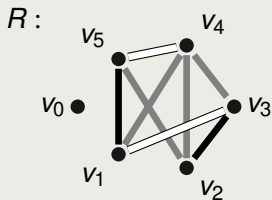
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- of choices for edges in white or black edges of R ,



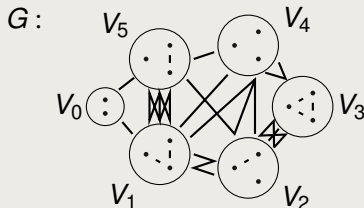
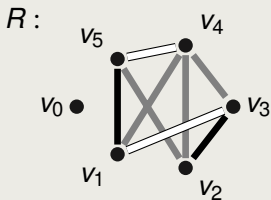
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- of choices for edges inside clusters,
- of choices for edges in irregular pairs and to V_0 ,
- of choices for edges in white or black edges of R ,
- of choices for edges in **grey edges** of R .



Step 3:

Lemma

$$|\mathcal{R}(R, \varepsilon, n, c)| \leq 2^{(f(\varepsilon, k_0) + h(c)) \binom{n}{2} + o(1) \binom{n}{2}}$$

And $f(\varepsilon, k_0) \rightarrow 0$ as $\varepsilon, \frac{1}{k_0} \rightarrow 0$.

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Structure of R

As any grey triangle would force an induced C_5 :

- At most half of the edges in E_R are grey.
- The choices in grey edges are at most $2^{h(c) \binom{n}{2} + o(1) \binom{n}{2}}$.

Step 4:

Lemma

Let G be a graph on n vertices. Then one of the following is true:

- there is a homogenous set of size $n/6$ in G , or
- G has $n/6$ disjoint copies of P_3 , or
- G has $n/6$ disjoint copies of $\overline{P_3}$.

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- if G has few disjoint copies of P_3 there is a large subgraph $G' \subseteq G$ with $G' \in \text{FORB}(P_3)$
 - G' is the disjoint union of cliques and thus has
 - a large homogenous set, or
 - many disjoint copies of $\overline{P_3}$.

Step 5:

Lemma

The number of choices for edges in grey pairs of the coloured reduced graph that do not give an induced C_5 is bounded by

$$2^{(h(c)-r(c))\binom{n}{2}+o(1)\binom{n}{2}}.$$

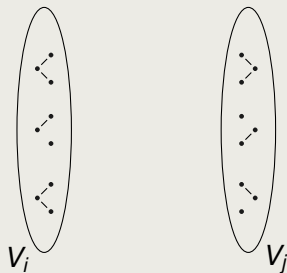
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- $\Theta(n^2)$ pairs of $P_3, \overline{P_3}$ in each grey edge



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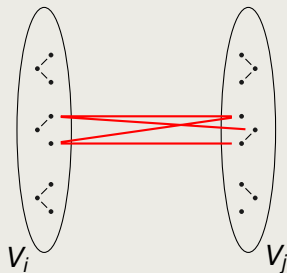
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- will select edges for grey pairs randomly
- $\Theta(n^2)$ pairs of $P_3, \overline{P_3}$ in each grey edge
- each with a positive probability to form an induced C_5
- global probability to have no induced C_5 :

$$2^{-r(c)\binom{n}{2}}$$



Step 5:

Finally...

- recall $|\mathcal{R}(R, \varepsilon, n, \mathbf{c})| \leq 2^{(f(\varepsilon, k_0) + h(\mathbf{c}))\binom{n}{2} + o(1)\binom{n}{2}}$

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- $|\mathcal{R}(R, \varepsilon, n, \mathbf{c}) \cap \text{FORB}_{n, \eta}^*(\mathbf{C}_5, \mathbf{c})| \leq 2^{(f(\varepsilon, k_0) + h(\mathbf{c}) - r(\mathbf{c})) \binom{n}{2} + o(1) \binom{n}{2}}$

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- $\frac{|\text{FORB}_{n, \eta}^*(C_5, c)|}{|\text{FORB}_n^*(C_5, c)|} \leq \frac{2^{(h(c) - 0.5r(c)) \binom{n}{2} + o(1) \binom{n}{2}}}{2^{h(c) \binom{n}{2} + o(1) \binom{n}{2}}} \rightarrow 0$ as $n \rightarrow \infty$

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- η only depends on the number of clusters

What's next?

- Does this method work for other graphs H ?
- Are there better bounds for $\eta(H)$?

Bibliography



Loebl, Reed, Scott, Thomason, Thomassé

Almost all H -free graphs have the Erdős-Hajnal property

submitted, 2010