Applied Geometry
and
Discrete Mathematics

Large Planar Subgraphs in Dense Graphs

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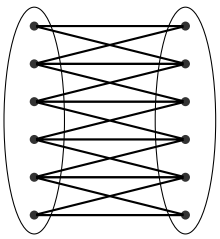
For a simple, undirected graph G let

$$\mathrm{pl}(G) = \max\{e(H) : H \subseteq G, H \text{ planar}\}.$$

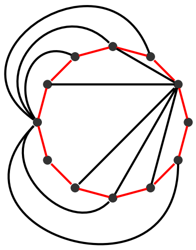
We call this parameter the *planarity* of G. Further let

$$\mathrm{pl}(n,d) = \min\{\mathrm{pl}(G) : |G| = n, \delta(G) \geq d\}\,.$$

Example: (for n even)



$\mathrm{pl}(n, n/2) \leq 2n - 4$
by Euler's Formula



$\mathrm{pl}(n, n/2) \geq 2n - 4$
by Dirac's Theorem

Previous Results

Kühn, Osthus, Taraz [2005]:

- $\forall \gamma > 0\, \exists\, C$ s.t. $\mathrm{pl}(n, \gamma n) \geq 2n - C$
- $\forall \gamma > 0\, \exists\, C$ s.t. $\mathrm{pl}(n, (1/2+\gamma)n) \geq 3n - C$

Kühn, Osthus [2005]:

- $\mathrm{pl}(n, 2/3n) = 3n - 6$

for n large enough

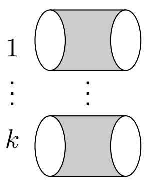
Constants for $d \leq n/2$

Allen, Skokan, W [2012+]:
For every $k \geq 2, \varepsilon > 0$ there is n_0 such that for all $n \geq n_0$

$$\mathrm{pl}\left(n, (\tfrac{1}{2k} + \varepsilon)n\right) \geq 2n - 4(k-1)\,,$$
$$\mathrm{pl}\left(n, \tfrac{1}{2k}n\right) \leq 2n - 4k\,.$$

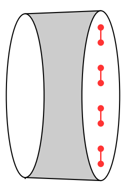
$\rightarrow (i)$

(i):



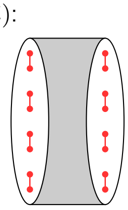
$\mathrm{pl}(n, \frac{n}{2k})$
$\leq 2n - 4k$

(ii):



$\mathrm{pl}(n, \frac{n+1}{2})$
$\leq (2{+}0.25)n - 4$

(iii):



$\mathrm{pl}(n, \frac{n+2}{2})$
$\leq (2{+}0.5)n - 4$

Upper bounds for $\mathrm{pl}(n,d)$

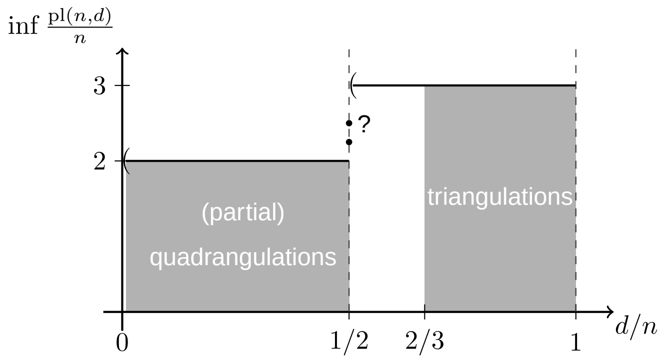
The Threshold at $d = n/2$

Łuczak, Taraz, W [2012+]:
For every $\varepsilon > 0$ there is n_0 such that for all $n \geq n_0$

$$\mathrm{pl}(n, (n+1)/2) \geq (2.25 - \varepsilon)n\,.$$

Furthermore $\mathrm{pl}(n, (n+1)/2) \leq 2.25n$ for n odd, $\rightarrow (ii)$

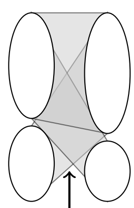
$$\mathrm{pl}(n, n/2 + n^{1/k}) \leq 2.5n + 2n/(k-2).$$



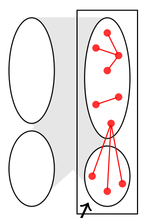
Minimum degree vs. planarity: A **jump** occurs at $d = n/2$.

Sketch of Proof / Tools

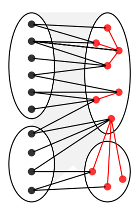
1. Apply Regularity Lemma to partition G.
2. Find a spanning star forest in the right side.
3. Use high density to construct planar $H \subseteq G$.



dense pairs



min. deg ≥ 1



planar subgraph

Conjecture: A 2nd Jump

For every $\varepsilon > 0$ there is n_0 such that for all $n \geq n_0$

$$\mathrm{pl}(n, (n+2)/2) \geq (2.5 - \varepsilon)n\,.$$

$\rightarrow (iii)$

Future Work

- Prove that $\mathrm{pl}(n, (n+2)/2) \geq (2.5 - \varepsilon)n$.
- Find an elementary proof for the threshold.
- Generalise $\mathrm{pl}(n,d)$ to other classes of monotone properties.